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Finite Element Analysis

Frame Equations
Axial Effects

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Lesson Outcomes

- At the end of this lesson, the student should be able to:
  - Develop frame equations including axial effects
  - Generate element stiffness matrix for a beam-column element

Beam Arbitrarily Oriented in Plane

- In the previous lecture, we developed element equations for a beam element arbitrarily orientated in 2D plane
- $[k] =$

$$\frac{EI}{L^3}\begin{bmatrix} 12S^2 & -12SC & -6LS & -12S^2 & 12SC & -6LS \\ -12SC & 12C^2 & 6LC & 12SC & -12C^2 & 6LC \\ -6LS & 6LC & 4L^2 & 6LS & -6LC & 2L^2 \\ -12S^2 & 12SC & 6LS & 12S^2 & -12SC & 6LS \\ 12SC & -12C^2 & -6LC & -12SC & 12C^2 & -6LC \\ -6LS & 6LC & 2L^2 & 6LS & -6LC & 4L^2 \end{bmatrix}$$

- This beam element includes shear and bending effects only
- A structural member resisting axial forces as well as shear and bending is known as beam-column

Beam-Columns

- Beam-columns are finite element models that can be used to analyze any structural member that resists axial forces in addition to shear and bending
- These are general prismatic elements
- It means that if any one of the effects is missing, these elements can still be used
- For example, if there is no bending, these elements can still model the column
- To develop element equations for beam-columns, we will include axial effects into the beam equations

Axial Effects

- We recall from bar element formulation that:
- $$\begin{Bmatrix} f_1' \\ f_2' \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix}$$
- Combining these axial effects with the shear and flexural effects of the beam stiffness matrix, we get the local stiffness matrix for a beam element including axial effects
- Recalling the local stiffness matrix for abeam element:
- $$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Axial Effects (Continued)

- Combining these two sets of equations:

- $$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m'_1 \\ f'_{2x} \\ f'_{2y} \\ m'_2 \end{Bmatrix} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} u'_1 \\ v'_1 \\ \phi'_1 \\ u'_2 \\ v'_2 \\ \phi'_2 \end{Bmatrix}$$

Axial Effects in Local Coordinates System

- From above, we can extract the stiffness matrix including the axial effects in local coordinates system as:

- $$[k'] = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

Transformation from Local to Global Coordinates System

- Since the axial effects will always be considered along the main axis of the member
- Therefore, for arbitrarily orientated members, we can include the transformation of element equations by adding ‘1’ in the diagonal position of the transformation matrix while retaining all the other values in the corresponding rows and columns as zero
- This results in a transformation matrix as given below:

Transformation from Local to Global Coordinates System (Continued)

- $$\begin{Bmatrix} u'_1 \\ v'_1 \\ \phi'_1 \\ u'_2 \\ v'_2 \\ \phi'_2 \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$
- From the above, we get transformation matrix $[T]$ as highlighted
- The global stiffness matrix can now be calculated as $[k] = [T]^T[k'][T]$

Global Stiffness Matrix for a Beam-Column

- After performing the necessary matrix operations, we get:
- $[k] =$

$$\frac{E}{L}\begin{bmatrix} AC^2+\frac{12I}{L^2}S^2 & \left(A-\frac{12I}{L^2}\right)CS & -\frac{6I}{L}S & -\left(AC^2+\frac{12I}{L^2}S^2\right) & -\left(A-\frac{12I}{L^2}\right)CS & -\frac{6I}{L}S \\ \left(A-\frac{12I}{L^2}\right)CS & AS^2+\frac{12I}{L^2}C^2 & \frac{6I}{L}C & -\left(A-\frac{12I}{L^2}\right)CS & -\left(AS^2+\frac{12I}{L^2}C^2\right) & \frac{6I}{L}C \\ -\frac{6I}{L}S & \frac{6I}{L}C & 4I & \frac{6I}{L}S & -\frac{6I}{L}C & 2I \\ -\left(AC^2+\frac{12I}{L^2}S^2\right) & -\left(A-\frac{12I}{L^2}\right)CS & \frac{6I}{L}S & \left(AC^2+\frac{12I}{L^2}S^2\right) & \left(A-\frac{12I}{L^2}\right)CS & \frac{6I}{L}S \\ -\left(A-\frac{12I}{L^2}\right)CS & -\left(AS^2+\frac{12I}{L^2}C^2\right) & -\frac{6I}{L}C & \left(A-\frac{12I}{L^2}\right)CS & \left(AS^2+\frac{12I}{L^2}C^2\right) & -\frac{6I}{L}C \\ -\frac{6I}{L}S & \frac{6I}{L}C & 2I & \frac{6I}{L}S & -\frac{6I}{L}C & 4I \end{bmatrix}$$

- This is the stiffness matrix for a beam-column arbitrarily orientated in plane and including axial effects
- An example employing this matrix will be studied in the next lecture

Author Information

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