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Finite Element Analysis

Arbitrarily Oriented Bar Element in 2 Dimensions

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Lesson Outcomes

- At the end of this lesson, the student should be able to:
  - Understand the concept of local and global coordinate systems
  - Transform the element equations of a bar element from local to global coordinate systems

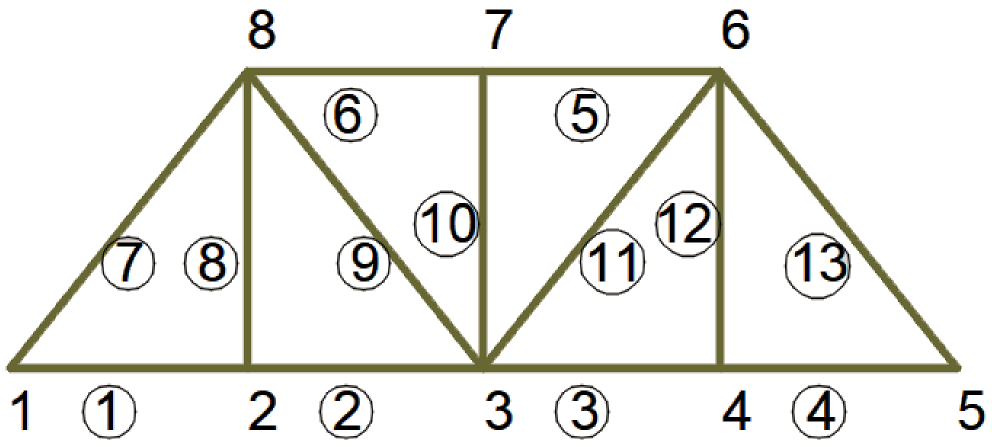
Stiffness Matrix of a Bar Element

- Stiffness Matrix for a bar element was developed in the previous lecture
- $[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
- This stiffness matrix is applicable to any bar element oriented in such a way that the x-axis is directed from node 1 to 2
- It is not the case always because bar elements are used in trusses, where some of these elements are vertical while others are inclined

Arbitrary Orientation of Bar Elements

- In the example truss shown below in Figure 1
- Elements 7 to 13 are not horizontal whereas elements 5 and 6 are oriented in the opposite direction to elements 1 to 4

Figure 1



Arbitrary Orientation of Bar Elements (Continued)

- Generally, when elements are not connected to consecutive nodes, the element is assumed to be orientated from the lower node number to the higher node number
- Therefore, the local x-axis of each element is orientated going from its lower node number to the higher node number
- The global x-axis is for the entire structure and is not affected by the orientation of individual elements

Arbitrary Orientation of Bar Elements (Continued)

- Figure below shows global x-axis and local x-axes for all the elements connected to node 3
- Superscripts indicate element numbers

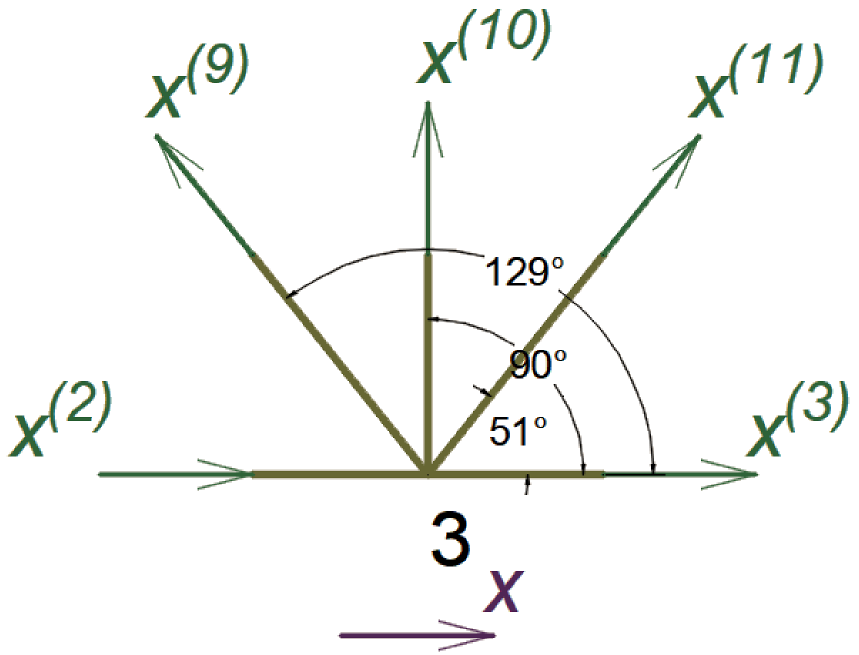


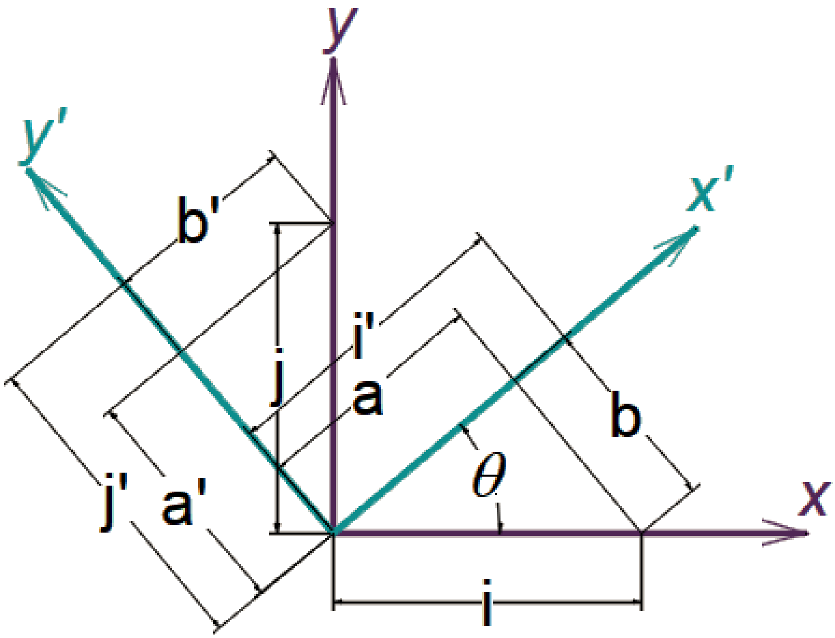
Figure 2

Arbitrary Orientation of Bar Elements (Continued)

- In order to use the stiffness matrix for the bar element developed earlier, the same will need to be transformed from the local to global coordinate systems
- After transformation, the stiffness matrix will be applicable to a bar element arbitrarily oriented in any direction in the x-y plane
- The transformation between a local coordinate system and a global coordinate system with angle θ between the two systems is described next

Transformation in 2 Dimensions

- Say a vector is arbitrarily oriented
- $\mathbf{d} = u\mathbf{i} + v\mathbf{j} = u'\mathbf{i}' + v'\mathbf{j}$
- Using vector addition
- $\mathbf{a} + \mathbf{b} = \mathbf{i}$
- $|\mathbf{a}| = |\mathbf{i}| \cos\theta$
- $\mathbf{i}$ is a unit vector; $\therefore\ |\mathbf{i}| =$
- $|\mathbf{a}| = \cos\theta$
- $|\mathbf{b}| = \sin\theta$



Transformation in 2 Dimensions (Continued)

- $\mathbf{a} = |\mathbf{a}|\mathbf{i}' = (\cos\theta)\mathbf{i}'$
- $\mathbf{b} = |\mathbf{b}|(-\mathbf{j}') = (\sin\theta)(-\mathbf{j}')$
- $\mathbf{i} = \cos\theta\ \mathbf{i}' - \sin\theta\ \mathbf{j}'$
- $\mathbf{a}' + \mathbf{b}' = \mathbf{j}$
- $\mathbf{a}' = (\cos\theta)\mathbf{j}'$
- $\mathbf{b}' = \sin\theta\ \mathbf{i}'$
- $\mathbf{j} = \sin\theta\ \mathbf{i}' + \cos\theta\ \mathbf{j}'$

Transformation in 2 Dimensions (Continued)

- $u(\cos\theta\,\mathbf{i}' - \sin\theta\,\mathbf{j}') + v(\sin\theta\,\mathbf{i}' + \cos\theta\,\mathbf{j}') = u'\mathbf{i}' + v'\mathbf{j}'$
- $u\cos\theta + v\sin\theta = u'$
- $-u\sin\theta + v\cos\theta = v'$
- In matrix form we can write as:
- $\begin{Bmatrix} u' \\ v' \end{Bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$
- $C = \cos\theta$ and $S = \sin\theta$

Transformation in 2 Dimensions (Continued)

- $\{d'\} = [T]\{d\}$
- $\{d\} = \begin{Bmatrix} u \\ v \end{Bmatrix}$
- $\{d'\} = \begin{Bmatrix} u' \\ v' \end{Bmatrix}$
- $[T] = \begin{bmatrix} C & S \\ -S & C \end{bmatrix}$
- This $[T]$ is known as the transformation matrix. It can be used to make any transformation between two coordinate systems

Global Stiffness Matrix for a Bar Element

- The nodal deformations of an arbitrarily orientated element will have components in global x and y directions
- These components are referred to as u and v
- An arbitrarily orientated bar element, therefore, will have 2 Degrees of Freedom (DOF) per node
- Using the transformation matrix developed, we can write the local element nodal deformations into global nodal deformations as:
- $\begin{Bmatrix} u_1' \\ v_1' \end{Bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$
- And for the other end:
- $\begin{Bmatrix} u_2' \\ v_2' \end{Bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}$

Global Stiffness Matrix for a Bar Element (Continued)

- For both ends of the element, therefore:

- $$\begin{Bmatrix} u_1' \\ v_1' \\ u_2' \\ v_2' \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

- Here, the transformation matrix is obtained as:

- $$[T] = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix}$$

Global Stiffness Matrix for a Bar Element (Continued)

- $\{d'\} = [T]\{d\}$
- $\{f'\} = [T]\{f\}$
- $\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ f'_{2x} \\ f'_{2y} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix}$
- $\{f'\} = [k']\{d'\}$
- $[T]\{f\} = [k'][T]\{d\}$
- $\{f\} = [T]^T[k'][T]\{d\}$
- $[k] = [T]^T[k'][T]$

Global Stiffness Matrix for a Bar Element (Continued)

- After performing the matrix multiplication:
- $$[k] = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$
- This is the global stiffness matrix for any bar element arbitrarily orientated in any direction
- Note that for an element orientated in positive x-direction, $\theta = 0, C = 1, S = 0$
- And for an element orientated in positive y-direction, $\theta = 90°, C = 0, S = 1$

Author Information

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