FACULTY OF INDUSTRIAL SCIENCES & TECHNOLOGY
FINAL EXAMINATION

COURSE	**:**	**NUMERICAL METHODS**
COURSE CODE	**:**	**BUM2313**
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DATE	**:**	**31 DECEMBER 2014**
DURATION	**:**	**3 HOURS**
SESSION/SEMESTER	**:**	**SESSION 2014/2015 SEMESTER I**
PROGRAMME CODE	**:**	**BEE/BEP/BEC/BMM/BMF/BMA/BMI/BMB/** **BAA/BAE/BFF/BFM**

INSTRUCTIONS TO CANDIDATES

1. This question paper consists of **SIX (6)** questions. Answer **ALL** questions.
2. Use **FOUR (4)** decimal places in all calculations.
3. All answers to a new question should start on new page.
4. All the calculations and assumptions must be clearly stated.
5. Candidates are not allowed to bring any material other than those allowed by the invigilator into the examination room.

EXAMINATION REQUIREMENT

1. Scientific calculator.

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO

This examination paper consists of **NINE (9)** printed pages including front page.

QUESTION 1

A system of two equations describing the intersection of a circle and an ellipse are given as follows

$$(x-4)^2+(y-1)^2=25$$
$$4(x-1)^2+16(y+3)^2=64$$

Find the points of intersection of this two curves using first iteration of Newton-Raphson method with initial estimates of $x(0)=0.5$ and $y(0)=0.5$.

(CO2, P01/11 Marks)

QUESTION 2

A liquid-liquid extraction process conducted in the Electrochemical Materials Laboratory involved the extraction of nickel (Ni) from the aqueous phase into an organic phase. A typical set of experimental data from the laboratory is given below

Observation	Ni aqueous phase, x (g/l)	Ni organic phase, $f(x)$ (g/l)
1	2	8.57
2	2.5	10.23
3	3	12.56
4	3.5	16.22
5	4	18.36
6	4.5	20.68

By assuming that x is the amount of Ni in the aqueous phase and $f(x)$ is the amount of Ni in organic phase for the above data

i. Estimate the Ni in organic phase using **second order Lagrange interpolating polynomial** if Ni in aqueous phase is 4.13g/l.

ii. Employ **second order Newton inverse interpolation polynomial** to determine the value of Ni in aqueous phase that correspond to 9.3g/l Ni in organic phase.

[***Hint***: *Choose the sequence of the data for your estimates to attain the best possible accuracy*]

(CO2, PO2/17 Marks)

QUESTION 3

Given

$$f(x) = \sin(x) \text{ and } g(x) = \sqrt{9 + x^2}$$

Find $\int_0^{\pi} 1 + f(x)g(x)dx$ by using

(a) Trapezoidal rule

(b) Trapezoidal rule with $n = 8$

(c) Simpson's 1/3 rule with $n = 8$

(d) Simpson's 3/8 rule

[***Hint*** : $\pi = 3.142$]

(CO2, PO1/13 Marks)

QUESTION 4

In the Lotka - Volterra model, under the assumption that the prey, x, learn to avoid the predators, y, the growth and decay rates due to predation will depend on the independent variable, t can be represented as

$$\frac{dx}{dt} = ax - \frac{b}{\left(e^t\right)^2} xy, \qquad x(0) = 4$$

$$\frac{dy}{dt} = -ry + \frac{c}{\left(e^t\right)^2} xy, \qquad y(0) = 2$$

where $a = 3, b = 2.4, c = 1.3$ and $r = 8.7$. Find $x(2)$ and $y(2)$ using fourth order Runge-Kutta method with step size, $\Delta t = 2$.

(CO2, PO1/17 Marks)

QUESTION 5

Consider a simple second order differential equation

$$6\frac{d^2x}{dt^2}-4\frac{dx}{dt}-2x=6t$$

with the boundary conditions $x(0)=10$, $x(10)=560$ and $h=5$. Use linear Shooting method to solve the problem with first initial guess $z(0)=-20$ and second initial guess $z(0)=20$.

(CO2, PO2/20 Marks)

QUESTION 6

If a cable uniform cross-section is suspended between two supports, the cable will sag forming a curved called a catenary. If we assume the lowest point on the curve lie on the *y*-axis, a distance y_0 above the origin, the differential equation governing is

$$\frac{d^2y}{dx^2}=\frac{1}{a}\left(y\sqrt{x}+\frac{dy}{dx}+x^3\right)$$

with boundary condition $y(0)=y_0$, $y(m)=120$.

(a) Given $a=9, m=20$ and $y_0=15$. Reduce the above boundary value problem to a tridiagonal system by using finite difference method with a step size, $\Delta x=4$.

(b) Solve the tridiagonal system in (a) by using Thomas algorithm method.

(CO2, PO2/22 Marks)

END OF QUESTION PAPER

APPENDIX

Chapter 1: Errors	
True Error E_t = true value - approximation value	**True percent relative error** $\varepsilon_t = \left\lvert \dfrac{\text{true value} - \text{approximation value}}{\text{true value}} \right\rvert \times 100$
Approximate percent relative error $\varepsilon_a = \left\lvert \dfrac{\text{present approximation} - \text{previous approximation}}{\text{present approximation}} \right\rvert \times 100$	**Stopping criterion** Terminate computation when $\lvert \varepsilon_a \rvert < \varepsilon_s$
Chapter 2: Roots of Equations	
Bisection method $x_r = \dfrac{(x_l + x_u)}{2}$	**False-position method** $x_r = x_u - \dfrac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$
Secant method $x_{i+1} = x_i - \dfrac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$	**Newton-Raphson method** $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

Chapter 3: Linear Algebraic Equations and Matrices

System of linear algebraic equations

$[A]\{X\} = \{B\}$. Decomposition $[A] = [L][U]$ with $[L]$ and $[U]$ can be obtained as follows:

Using Doolittle decomposition

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}; [U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Using Cholesky method

$$[A] = [U]^T [U] \; ; \quad [U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix},$$

$$u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2}$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj}}{u_{ii}} \quad \text{for } j = i+1, \ldots, n$$

Using Crout's method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Jacobi method

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \left(\sum_{\substack{j=1 \\ j\neq i}}^{j=n} a_{ij} x_j^{(k)} \right) \right], i = 1, 2, ..., n.$$

Gauss seidel method

$$x_i^{(k+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{(k+1)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(k)}$$

where

$k = 1, 2, \ldots$

$i = 1, 2, \ldots, n$

Power method

$$v^{(k+1)} = \frac{1}{m_{k+1}} A v^{(k)}$$

$k = 0,1,2,\ldots.$

Nonlinear system: Newton-Raphson method

$$x_{1,i+1} = x_{1,i} - \frac{f_{1,i}\dfrac{\partial f_{2,i}}{\partial x_2} - f_{2,i}\dfrac{\partial f_{1,i}}{\partial x_2}}{|J|}$$

$$x_{2,i+1} = x_{2,i} - \frac{f_{2,i}\dfrac{\partial f_{1,i}}{\partial x_1} - f_{1,i}\dfrac{\partial f_{2,i}}{\partial x_1}}{|J|}$$

$$[J] = \begin{bmatrix} \dfrac{\partial f_{1,i}}{\partial x_1} & \dfrac{\partial f_{1,i}}{\partial x_2} \\ \dfrac{\partial f_{2,i}}{\partial x_1} & \dfrac{\partial f_{2,i}}{\partial x_2} \end{bmatrix}$$

Chapter 4: Curve Fitting

Newton interpolation polynomial

$$f_n(x) = f(x_0) + f[x_0, x_1](x - x_0)$$
$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ \cdots + f[x_0, \ldots, x_{n-1}, x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

where

$$f[x_0, \ldots, x_{n-1}, x_n] = \frac{f[x_n, x_{n-1}, \ldots, x_1] - f[x_{n-1}, x_{n-2}, \ldots, x_0]}{x_n - x_0}$$

Lagrange interpolation polynomial

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i) \text{ where } L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$n -$ order of interpolation

Inverse Newton interpolation polynomial

$$P_n(f) = x_0 + b_1(f - f_0) + b_2(f - f_0)(f - f_1) + b_3(f - f_0)(f - f_1)(f - f_2) + \cdots + b_n(f - f_0)(f - f_1)\cdots(f - f_{n-1})$$

$n -$ order of interpolation

Inverse Lagrange interpolation polynomial

$$P_n(f) = \sum_{i=0}^{n} L_i(f)x_i \quad \text{where} \quad L_i(f) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(f - f_j)}{(f_i - f_j)}$$

$n -$ order of interpolation

Linear Splines

$$s_i(x) = f(x_i) + \frac{f_{i+1} - f_i}{x_{i+1} - x_i}(x - x_i) \qquad x_i \le x \le x_{i+1}$$

Quadratic Splines

$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 \qquad x_i \le x \le x_{i+1}$$

For $i = 1, 2, \ldots, n-1$, find

$h_i = x_{i+1} - x_i$; $\qquad f_i + b_i h_i + c_i h_i^2 = f_{i+1}$

$b_i + 2c_i h_i = b_{i+1}$;

Also given,

$c_1 = 0$

$a_i = f_i$

Chapter 5: Numerical Integration

Trapezoidal rule

$$I \cong \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right]$$

where

$$h = \frac{x_n - x_0}{n}$$

Simpson's 1/3rd rule

$$I \cong \frac{h}{3}\left[f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n) \right]$$

where

$h = \dfrac{x_n - x_0}{n}$ and n must even segment

Simpson's 3/8 rule

$$I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right], \text{ where } h = \frac{x_3 - x_0}{3}$$

Chapter 6: Ordinary Differential Equations (IVP)

Euler's method

$$y_{i+1} = y_i + hf(x_i, y_i)$$
$$x_{i+1} = x_i + h$$

2^{nd} order Runge-Kutta: Heun method	**2^{nd} order Runge-Kutta: Midpoint method**
$y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)h$ $x_{i+1} = x_i + h$ where $k_1 = f(x_i, y_i)$ $k_2 = f(x_i + h, y_i + k_1 h)$	$y_{i+1} = y_i + k_2 h$ $x_{i+1} = x_i + h$ where $k_1 = f(x_i, y_i)$ $k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$
2^{nd} order Runge-Kutta: Ralston's method $y_{i+1} = y_i + \frac{1}{3}(k_1 + 2k_2)h$ $x_{i+1} = x_i + h$ where $k_1 = f(x_i, y_i)$ $k_2 = f(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h)$	
Fourth order Runge-Kutta method $y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$ $x_{i+1} = x_i + h$ where $k_1 = f\left(x_i, y_i\right)$ $k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$ $k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$ $k_4 = f\left(x_i + h, y_i + k_3 h\right)$	

Chapter 6: Ordinary Differential Equations (BVP)

Shooting method

Extrapolate estimate for initial slope

$$z(0) = G1 + \frac{G2 - G1}{R2 - R1}(D - R1)$$

where

G1 = First guess at initial slope
G2 = Second guess at initial slope
R1 = Final result at endpoint (using G1)
R2 = Second result at endpoint (using G2)
D = the desired value at the endpoint

Finite Difference method

$$\frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}$$

$$\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2\Delta x}$$

$$\left(1 - \frac{\Delta x}{2} p_i\right) y_{i-1} - \left(2 - \Delta x^2 q_i\right) y_i + \left(1 + \frac{\Delta x}{2} p_i\right) y_{i+1} = \Delta x^2 r_i$$

Thomas Algorithm

$$\alpha_i = d_i - c_i \beta_{i-1} \quad , \quad \alpha_1 = d_1$$

$$\beta_i = \frac{e_i}{\alpha_i}$$

$$w_i = \frac{b_i - c_i w_{i-1}}{\alpha_i} \quad , \quad w_1 = \frac{b_1}{\alpha_1}$$

$$y_i = w_i - \beta_i y_{i+1} \quad , \quad y_4 = w_4$$