FACULTY OF INDUSTRIAL SCIENCES & TECHNOLOGY
FINAL EXAMINATION

COURSE	**:**	**NUMERICAL METHODS**
COURSE CODE	**:**	**BUM2313**
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PROGRAMME CODE	**:**	**BEE/BEP/BEC/BMM/BMF/BMA/BMI/BMB/ BAA/BAE**

INSTRUCTIONS TO CANDIDATES

1. This question paper consists of **SIX (6)** questions. Answer **ALL** questions.
2. Use **FOUR (4)** decimal places in all calculations.
3. All answers to a new question should start on new page.
4. All the calculations and assumptions must be clearly stated.
5. Candidates are not allowed to bring any material other than those allowed by the invigilator into the examination room.

EXAMINATION REQUIREMENTS

1. APPENDIX
2. Scientific calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO

This examination paper consists of **TEN(10)** printed pages including front page.

QUESTION 1

Given the system of linear equations

$$20x_1 + 12.5x_2 = 76.2 - 16.4x_3$$
$$2.5x_1 + 2.2x_3 - 58.4 = -5x_2$$
$$6x_1 + 3.3x_2 + 8x_3 - 62.11 = 0.$$

(i) Transform the above system of linear equations in matrix form, $AX = b$.

(ii) Decompose matrix A into lower and upper triangular matrix using Crout's method.

(iii) Solve the system of linear equations.

(CO2,PO1/20 Marks)

QUESTION 2

The growth rate of bacteria, k (mg/L) with respect to oxygen concentration, c (mg/L) can be modelled by the following equation

$$k = \frac{k_{\max} c^2}{c_s + c^2}$$

where c_s and $k_{\max}$ are parameters. An experiment to determine the growth rate of bacteria as a function of oxygen concentration was conducted. The result of the experiment is in **Table 1**.

Table 1

c (mg/L)	0.5	2.0	4.0	6.5
k (mg/L)	2.2	6.6	4.7	8.1

(i) Use the quadratic splines interpolation to fit the given data.

(ii) Estimate the growth rate of bacteria at oxygen concentration of 3.7mg/L.

(CO2,PO2/15 Marks)

QUESTION 3

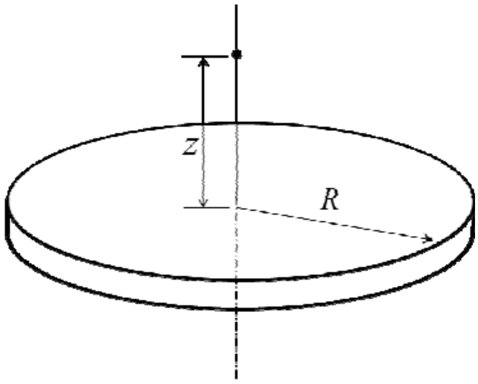

Figure 1

The electric field, E due to a charged circular disk at a point with a distance z along the axis of the disk as depicted in **Figure 1** is given by

$$E = \frac{\sigma z}{4\varepsilon_0} \int_0^R 2r\left(z^2 + r^2\right)^{-\frac{3}{2}} dr$$

where the charge density, $\sigma = 300\ \mu\text{C/cm}^2$, the permittivity constant, $\varepsilon_0 = 8.85 \times 10^{-2}\ \text{C}^2/\text{N-cm}^2$, and the radius of the disk, $R = 60\,\text{cm}$.

(i) Determine the electric field at a point with a distance $5\,\text{cm}$ using Trapezoidal rule method with $n = 8$.

(ii) Calculate the true percent relative error if the exact value of electric field is $1554.1602\ N/C$.

(CO2, PO1/10 Marks)

QUESTION 4

The rate of heat flow between two points on a heated cylinder at one end is given by

$$\frac{dQ}{dt} = \lambda A\left(\frac{100(L-x)(20-t)}{100-xt}\right)$$

where $\lambda = 0.4\ \text{cal}\cdot\text{cm/s}$ is a constant, $A = 10\ \text{cm}^2$ represent the cylinder's cross-sectional area, $L = 20\ \text{cm}$ is the length of the rod, $x = 2.5\ \text{cm}$ is the distance from the heated end and $Q(0) = 0$ is the initial condition of heat flow at $t_0 = 0$. Compute the heat flow for $0 \le t \le 6$ by using

(i) Second order Runge-Kutta of Heun method with a step size of 3; and

(ii) Fourth order Runge-Kutta method with a step size of 6.

(CO2,PO2/15 Marks)

QUESTION 5

The position, x of a falling object at time, t is governed by

$$2\frac{d^2x}{dt^2} = 9.81 - \frac{15.75}{90}\frac{dx}{dt}$$

with boundary conditions, $x(0) = 0$ and $x(20) = 500$. Use linear shooting method with Euler's approximation and $\Delta t = 10$ to obtain the solution for the above problem with the first initial guess, $z(0) = -10$ and second initial guess, $z(0) = 10$.

(CO2,PO2/18 Marks)

QUESTION 6

The temperature distribution in a tapered conical cooling fin is described by the differential equation

$$\frac{d^2u}{dx^2} + 2x^2\left(\frac{du}{dx}\right) + Pu - x = 0$$

where u is a temperature, x is an axial distance and P is a nondimensional parameter that describes the heat and geometry

$$P = \frac{hL}{k}\sqrt{1 + \frac{4}{2m^2}} \ .$$

The term h represents a heat coefficient, k is thermal conductivity, L is the length or height of the cone and m represent the slope of the cone wall. The equation has the boundary conditions $u(0) = 0$ and $u(1.25) = 1$.

(i) Let $h = 0.5$, $k = 0.2$, $L = 1$ and $m = 0.5$. Use a Finite Difference method with a step size of 0.25 to reduce the above boundary value problem to a tridiagonal system.

(ii) Solve the tridiagonal system in (i) using Thomas algorithm method.

(CO2,PO2/22 Marks)

END OF QUESTION PAPER

APPENDIX

Errors

True Error

E_t = true value - approximation value

True percent relative error

$$\varepsilon_t = \left| \frac{\text{true value} - \text{approximation value}}{\text{true value}} \right| \times 100$$

Approximate percent relative error

$$\varepsilon_a = \left| \frac{\text{present approximation} - \text{previous approximation}}{\text{present approximation}} \right| \times 100$$

Stopping criterion

Terminate computation when $|\varepsilon_a| < \varepsilon_s$

Roots of Equations

Bisection method

$$x_r = \frac{(x_l + x_u)}{2}$$

False-position method

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

Secant method

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

Newton-Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Linear Algebraic Equations and Matrices

System of linear algebraic equations

$[A]\{X\} = \{B\}$. Decomposition $[A] = [L][U]$ with $[L]$ and $[U]$ can be obtained as follows:

Using Doolittle decomposition

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}; [U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Using Cholesky method

$$[A] = [U]^T[U]\ ; \quad [U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix},$$

$$u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2}$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj}}{u_{ii}} \quad \text{for } j = i+1, \ldots, n$$

Using Crout's method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Jacobi method

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[b_i - \left(\sum_{\substack{j=1 \\ j \neq i}}^{j=n} a_{ij} x_j^{(k)} \right) \right], i = 1, 2, ..., n.$$

Gauss seidel method

$$x_i^{(k+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{(k+1)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(k)}$$

where

$k = 1, 2, \ldots$

$i = 1, 2, \ldots, n$

Power method

$$v^{(k+1)} = \frac{1}{m_{k+1}} A v^{(k)}$$

$k = 0,1,2,....$

Nonlinear system: Newton-Raphson method

$$x_{1,i+1} = x_{1,i} - \frac{f_{1,i} \frac{\partial f_{2,i}}{\partial x_2} - f_{2,i} \frac{\partial f_{1,i}}{\partial x_2}}{|J|}$$

$$x_{2,i+1} = x_{2,i} - \frac{f_{2,i} \frac{\partial f_{1,i}}{\partial x_1} - f_{1,i} \frac{\partial f_{2,i}}{\partial x_1}}{|J|}$$

$$[J] = \begin{bmatrix} \frac{\partial f_{1,i}}{\partial x_1} & \frac{\partial f_{1,i}}{\partial x_2} \\ \frac{\partial f_{2,i}}{\partial x_1} & \frac{\partial f_{2,i}}{\partial x_2} \end{bmatrix}$$

Curve Fitting

Newton interpolation polynomial

$$f_n(x) = f(x_0) + f[x_0, x_1](x - x_0)$$
$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ \cdots + f[x_0, \ldots, x_{n-1}, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

where

$$f[x_0, \ldots, x_{n-1}, x_n] = \frac{f[x_n, x_{n-1}, \ldots, x_1] - f[x_{n-1}, x_{n-2}, \ldots, x_0]}{x_n - x_0}$$

Lagrange interpolation polynomial

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i) \text{ where } L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$n -$ order of interpolation

Inverse Newton interpolation polynomial

$$P_n(f) = x_0 + b_1(f - f_0) + b_2(f - f_0)(f - f_1) + b_3(f - f_0)(f - f_1)(f - f_2) + \cdots + b_n(f - f_0)(f - f_1)\cdots(f - f_{n-1})$$

n – order of interpolation

Inverse Lagrange interpolation polynomial

$$P_n(f) = \sum_{i=0}^{n} L_i(f)x_i \quad \text{where} \quad L_i(f) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(f - f_j)}{(f_i - f_j)}$$

n – order of interpolation

Linear Splines

$$s_i(x) = f(x_i) + \frac{f_{i+1} - f_i}{x_{i+1} - x_i}(x - x_i) \qquad x_i \le x \le x_{i+1}$$

Quadratic Splines

$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 \qquad x_i \le x \le x_{i+1}$$

For $i = 1, 2, ..., n-1$, find

$h_i = x_{i+1} - x_i$; $\quad f_i + b_i h_i + c_i h_i^2 = f_{i+1}$

$b_i + 2c_i h_i = b_{i+1}$;

Also given,

$c_1 = 0$

$a_i = f_i$

Numerical Integration

Trapezoidal rule

$$I \cong \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right]$$

where

$$h = \frac{x_n - x_0}{n}$$

Simpson's 1/3rd rule

$$I \cong \frac{h}{3}\left[f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n) \right]$$

where

$h = \dfrac{x_n - x_0}{n}$ and n must even segment

Simpson's 3/8 rule

$$I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right], \text{ where } h = \frac{x_3 - x_0}{3}$$

Ordinary Differential Equations (IVP)

Euler's method

$$y_{i+1} = y_i + hf(x_i, y_i)$$

$$x_{i+1} = x_i + h$$

2nd order Runge-Kutta: Heun method

$$y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)h$$

$$x_{i+1} = x_i + h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

2nd order Runge-Kutta: Midpoint method

$$y_{i+1} = y_i + k_2 h$$

$$x_{i+1} = x_i + h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

2nd order Runge-Kutta: Ralston's method

$$y_{i+1} = y_i + \frac{1}{3}(k_1 + 2k_2)h$$

$$x_{i+1} = x_i + h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h)$$

Fourth order Runge-Kutta method

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$x_{i+1} = x_i + h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f\left(x_i + h, y_i + k_3 h\right)$$

Ordinary Differential Equations (BVP)

Shooting method

Extrapolate estimate for initial slope

$$z(0) = G1 + \frac{G2 - G1}{R2 - R1}(D - R1)$$

where

G1 = First guess at initial slope
G2 = Second guess at initial slope
R1 = Final result at endpoint (using G1)
R2 = Second result at endpoint (using G2)
D = the desired value at the endpoint

Finite Difference method

$$\frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}$$

$$\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2\Delta x}$$

$$\left(1 - \frac{\Delta x}{2} p_i\right) y_{i-1} - \left(2 - \Delta x^2 q_i\right) y_i + \left(1 + \frac{\Delta x}{2} p_i\right) y_{i+1} = \Delta x^2 r_i$$

Thomas Algorithm

$$\alpha_i = d_i - c_i \beta_{i-1} \quad , \quad \alpha_1 = d_1$$

$$\beta_i = \frac{e_i}{\alpha_i}$$

$$w_i = \frac{b_i - c_i w_{i-1}}{\alpha_i} \quad , \quad w_1 = \frac{b_1}{\alpha_1}$$

$$y_i = w_i - \beta_i y_{i+1} \quad , \quad y_4 = w_4$$