10. Ordinary Differential Equations: Boundary Value Problem

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10.1 Exercises

Shooting Method

Exercise 10.1 Approximate the solution of the following boundary value problems (BVP) by using Shooting method.

i. $y'' + y' - 2y = 2(x^2 - 4x + \sin(x))$, $y(0) = -0.6$, $y(1) = -0.8095$, $h = 0.2$.
Let the first guess $z_0 = 0.4$ and the second guess $z_0 = 1.0$.

ii. $y'' + y = 0$, $y(0) = 1$, $y(\pi) = 0$, $h = \frac{\pi}{4}$.
Let the first guess $z_0 = 1.5$ and the second guess $z_0 = 2.0$.

iii. $y'' - x = \left(1 - \frac{x}{5}\right)y$, $y(0) = 2.0$, $y(3) = -1.0$, $h = 0.5$.
Let the first guess $z_0 = 3.0$ and the second guess $z_0 = -3.0$.

iv. $y'' - y' - 2y - \cos(x) = 0$, $y(0) = -0.3$, $y\left(\frac{\pi}{2}\right) = -0.1$, $h = \frac{\pi}{4}$.
Let the first guess $z_0 = 3.0$ and the second guess $z_0 = -3.0$.

v. $y'' - \frac{y'}{x} = \frac{3}{x^2}y + \frac{\ln x}{x} - 1$, $y(1) = y(3) = 0$, $h = 0.5$.
Let the first guess $z_0 = 2.0$ and the second guess $z_0 = -2.0$.

vi. $y'' + 4y - \cos(x) = 0$, $y(0) = 0$, $y\left(\frac{\pi}{4}\right) = 0$, $h = \frac{\pi}{20}$
Let the first guess $z_0 = 0.5$ and the second guess $z_0 = -0.5$.

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Exercise 10.2 The position, x of a falling object at time, t is governed by

$$2\frac{d^2x}{dt^2} = 9.8 - 2\frac{dx}{dt}$$

with boundary conditions, $x(0) = 0$ and $x(20) = 800$. Approximate the solution of the psoition x for each interior nodes by using a linear Shooting method. Use the first initial guess, $z_0 = 5$ and second initial guess, $z_0 = -5$ and $h = 5$.
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Exercise 10.3 A boundary value problem for a temperature distribution, T of a non–insulated uniform rod is given by

$$\frac{d^2T}{dx^2} + kT = 0, \quad T(0) = 0, \quad T(20) = 10.$$

Suppose $k = 0.75$, approximate the interior temperature distribution, T using the Shooting method with a step size, $h = 4$. ■

Exercise 10.4 A simple supported beam at $x_0 = 0$ and $x_n = L$ with a uniform load w and the vertical deflection $v(x)$ is described by the boundary value problem

$$\frac{d^2v}{dx^2} = \left(\frac{x-L}{3EI}\right)wx, \quad v(0) = 5, \quad v(L) = 100$$

where L is the length of the beam, E represents the Young's modulus of the material from which the beam is fabricated and I is the second moment of area of the beam's cross–section. Suppose $L = 15$ m, $E = 210$ kN/m^2, $I = 370.5$ m^2 and $w = 9.8$ kN/m. Use the Shooting method to approximate the vertical deflection, $v(x)$ with the step size, $h = 3$, the first guess, $z_0 = 10$ and the second guess, $z_0 = 5$. ■

Exercise 10.5 The Van der Pol equation

$$y'' - \gamma(y^2 - 1)y' + y = 0, \quad \gamma > 0$$

governs the flow of current in a vacuum tube with three internal elements. Let $\gamma = \frac{1}{2}$, $y(0) = 0$ and $y(2) = 1$. Approximate the solution of $y(t)$, for $t = 0.2i$, where $1 \le i \le 9$. ■

10.1.1 Finite Difference Method

Exercise 10.6 Approximate the following boundary value problems using Finite difference method.

i. $y'' + xy' + x^2y = 2x^3, \quad y(0) = 1, \quad y(1) = -1, \quad h = 0.2.$

ii. $y'' - xy' = -3y + 11x, \quad y(0) = 1, \quad y(2) = -1, \quad h = 0.4.$

iii. $y'' - y' - 2y = 2\cos(x), \quad y(0) = -0.3, \quad y(1) = -0.1, \quad h = 0.2$

iv. $y'' - \frac{y'}{x} = \frac{3}{x^2}y + \frac{\ln x}{x} - 1, \quad y(1) = y(2) = 0, \quad n = 5$

v. $y'' + 4y - \cos(x) = 0, \quad y(0) = 0, \quad y\left(\frac{\pi}{4}\right) = 0, h = \frac{\pi}{20}$

Exercise 10.7 A heated rod with a uniform heat source can be modelled by the Poisson equation

$$\frac{d^2\theta}{dx^2} = -g(x)$$

where θ is a temperature distribution in the direction of heat flow, x denotes the local position with respect to x–coordinate and $g(x)$ is a heat source. Given $g(x) = 28$ and the boundary conditions $\theta(0) = 50$ and $\theta(20) = 400$, approximate the temperature distribution using the Finite Difference method. Use a step size, $\Delta x = 5$.

Exercise 10.8 A steady–state one–dimensional heat flow in a circular rod with internal heat source H over the range $1 \leq x \leq 2$ can be modelled by

$$x\frac{d^2\theta}{dx^2} + \frac{d\theta}{dx} = H$$

with boundary conditions $\theta(1) = 0$ and $\theta(2) = 20$. Use the Finite Difference method to approximate the heat flow at the local position, x if the internal heat source is $20x$ K/m^2 and $\Delta x = 0.2$.

Exercise 10.9 Consider a cylinder of radius, $R = 2$ m, with uniformly distributed heat sources and constant thermal conductivity, k. If cylinder is sufficiently long that the temperature, θ may be considered a function of radius only, the appropriate differential equation is

$$x\frac{d^2\theta}{dx^2} + \frac{d\theta}{dx} + x\frac{Q}{k} = 0.$$

where Q is heat generated per unit volume. The boundary conditions are $\theta(1) = 10$ and $\theta(R) = 215$ and heat generated equals heat lost at the surface such that

$$Q\pi R^2 L = -2\pi kRL$$

where $L = 1$ m is a length of the cylinder. Suppose $k = 2$ and a step size $\Delta x = 0.1$, approximate the temperature distribution via Finite Difference method.

Exercise 10.10 If a cable uniform cross-section is suspended between two supports, the cable will sag forming a curved called a catenary. If we assume the lowest point on the curve lie on the y–axis, the governing differential equation is

$$\frac{d^2y}{dx^2} = \frac{3a}{4}x - a\left(x - \frac{L}{4}\right)$$

with boundary conditions $y(0) = 0$ and $y(L) = 0$. Given a scaling parameter, $a = 2$, a cable length, $L = 20$ and $\Delta x = 4$, approximate the solution of the above BVP using the Finite Difference method.

References 1. Chapra, C. S. & Canale, R. P. Numerical Methods for Engineers, Sixth Edition, McGraw–Hill, 2010.