BUM2113 Ordinary Differential Equations

Chapter 3B: Laplace Transforms

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Chapter Description

Expected Outcomes

1.Determine the inverse Laplace Transforms using its properties

References

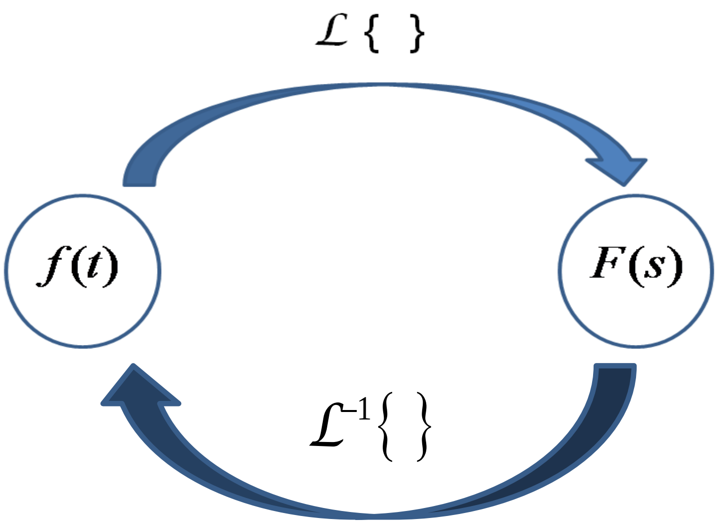
Samsudin Abdullah, Nadirah Nasir, Rahimah Jusoh @ Awang, Laila Amera Aziz, Wan Nur Syahidah Wan Yusoff,, Module : Ordinary Differential Equations (BUM2133), 4rd Edition 2016.

Content

3.3 Inverse Laplace Transform
3.4 The Convolution Theorem
3.5 Laplace Transform of Integral

3.3 INVERSE LAPLACE TRANSFORMS

$$\text{If } \mathcal{L}\{f(t)\} = F(s), \text{ then } f(t) = \mathcal{L}^{-1}\{F(s)\}$$



Laplace Inverse Table

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$F(s),\ s>0$	$f(t)$
$\frac{a}{s}$	a
$\frac{n!}{s^{n+1}}$	t^n
$\frac{1}{s-a}$	e^{at}
$\frac{a}{s^2+a^2}$	$\sin at$
$\frac{s}{s^2+a^2}$	$\cos at$

$F(s),\ s>0$	$f(t)$
$\frac{a}{s^2-a^2}$	$\cosh at$
$\frac{s}{s^2-a^2}$	$\sinh at$
$\frac{e^{-sd}}{s}$	$u(t-d)$
e^{-sd}	$\delta(t-d)$
1	$\delta(t)$

Example : Find the inverse Laplace transforms of

(a) $\dfrac{6}{s}$ (b) $\dfrac{1}{s^3}$

(c) $\dfrac{3}{2s-6}$ (d) $\dfrac{7}{s^2+3}$

(e) $\dfrac{s}{4s^2-9}$ (f) $\dfrac{2s-7}{s^2+16}$

Solution:

(a) $$\mathcal{L}^{-1}\left\{\frac{6}{s}\right\}=6$$

(b) $$\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}=\mathcal{L}^{-1}\left\{\frac{1\times 2!/2!}{s^{2+1}}\right\}=\frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\}=\frac{1}{2}t^2$$

(c) $$\mathcal{L}^{-1}\left\{\frac{3}{2s-6}\right\}=\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}=\frac{3}{2}e^{3t}$$

(d) $$\mathcal{L}^{-1}\left\{\frac{7}{s^2+3}\right\}=\mathcal{L}^{-1}\left\{\frac{7\times \sqrt{3}/\sqrt{3}}{s^2+\left(\sqrt{3}\right)^2}\right\}=\frac{7}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{s^2+\left(\sqrt{3}\right)^2}\right\}$$

$$=\frac{7}{\sqrt{3}}\sin\sqrt{3}t$$

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(e)
$$\mathcal{L}^{-1}\left\{\frac{s}{4s^2-9}\right\}=\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2-9/4}\right\}=\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2-\left(3/2\right)^2}\right\}$$
$$=\frac{1}{4}\cos\frac{3}{2}t$$

(f)
$$\mathcal{L}^{-1}\left\{\frac{2s-7}{s^2+16}\right\}=\mathcal{L}^{-1}\left\{\frac{2s}{s^2+4^2}\right\}-\mathcal{L}^{-1}\left\{\frac{7\times 4/4}{s^2+4^2}\right\}$$
$$=2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\}-\frac{7}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\}$$
$$=2\cos 4t-\frac{7}{4}\sin 4t$$

3.3.2 INVERSION OF FIRST SHIFT THEOREM

From the first shift theorem,

$$\text{If } \mathcal{L}\left\{e^{-at} f(t)\right\} = F(s+a) \text{ then } \mathcal{L}^{-1}\left\{F(s+a)\right\} = e^{-at} f(t)$$

$$\mathcal{L}^{-1}\left\{ F(s+a) \right\} = e^{-at} \quad f(t)$$

$$a = ?,\ F(s+a) \longrightarrow F(s) \xrightarrow{\mathcal{L}^{-1}\{\ \}} f(t)$$

Note : The value of *a* is ALWAYS identified from the denominator of the fraction.

Example : Use the first shift theorem to find the inverse Laplace transforms of the following functions.

(a) $$\frac{4}{(s-3)^3}$$

(b) $$\frac{s}{(s+4)^2+1}$$

(c) $$\frac{s-1}{s^2+4s+6}$$

Solution:

(a)

$$a = -3, \quad F(s-3) = \frac{4}{(s-3)^3}$$

$$F(s) = \frac{4}{s^3}$$

$(s-3) \rightarrow s$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{4}{s^3}\right\}$$

$$= 2t^2$$

$$\mathcal{L}^{-1}\left\{\frac{4}{(s-3)^3}\right\} = e^{3t}\,2t^2$$

(b)

$$a = 4, \quad F(s+4) = \frac{s}{(s+4)^2 + 1}$$

$$= \frac{(s+4) - 4}{(s+4)^2 + 1}$$

substitute $s = (s+4) - 4$ then convert $(s+4) \to s$

$$F(s) = \frac{s-4}{s^2+1}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s-4}{s^2+1}\right\}$$

$$= \cos t - 4\sin t$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+4)^2+1}\right\} = e^{-4t}\left(\cos t - 4\sin t\right)$$

(c) In order to identify a, use completing the square method.

$$\frac{s-1}{s^2+4s+6}=\frac{s-1}{(s+2)^2+2}$$

$$a=2,\quad F(s+2)=\frac{s-1}{(s+2)^2+2}$$

$$=\frac{(s+2)-3}{(s+2)^2+2}$$

$$F(s)=\frac{s-3}{s^2+2}$$

$$f(t)=\cos\sqrt{2}t-\frac{3}{\sqrt{2}}\sin\sqrt{2}t$$

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2+4s+6}\right\}=e^{-2t}\left(\cos\sqrt{2}t-\frac{3}{\sqrt{2}}\sin\sqrt{2}t\right)$$

3.3.3 INVERSION USING PARTIAL FRACTION

EXAMPLE OF PARTIAL FRACTION DECOMPOSITION:

- $$\frac{s^2-2s+3}{(s-2)(s+3)(s+5)}=\frac{A}{s-2}+\frac{B}{s+3}+\frac{C}{s+5}$$
- $$\frac{5s+3}{s^2(s+2)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+2}+\frac{D}{(s+2)^2}$$
- $$\frac{s+6}{(2s^2-5s)(2s^2+3s+4)}=\frac{A}{s}+\frac{B}{2s-5}+\frac{Cx+D}{2s^2+3s+4}$$

Example: Express the following expression as partial fraction and find the Laplace inverse.

(a) $$\frac{7s+3}{s^3+3s^2-4s}$$

(b) $$\frac{s^2+9s+22}{(s-1)(s+3)^2}$$

Solution :

(a) $$\frac{7s+3}{s(s-1)(s+4)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+4}$$

$$7s+3=A(s-1)(s+4)+Bs(s+4)+Cs(s-1)$$

Can either use substitution or comparison method

$$s=0: \quad 3=A(-1)(4) \Rightarrow A=-3/4$$

$$s=1: \quad 10=B(1)(5) \Rightarrow B=2$$

$$s=-4: -25=C(-4)(-5) \Rightarrow C=5/4$$

$$\mathcal{L}^{-1}\left\{\frac{7s+3}{s^3+3s^2-4s}\right\}=\mathcal{L}^{-1}\left\{\frac{-3/4}{s}+\frac{2}{s-1}+\frac{5/4}{s+4}\right\}$$

$$=-\frac{3}{4}+2e^t+\frac{5}{4}e^{-4t}$$

(b) $$\frac{s^2+9s+22}{(s-1)(s+3)^2}=\frac{A}{s-1}+\frac{B}{s+3}+\frac{C}{(s+3)^2}$$

$$A(s+3)^2+B(s-1)(s+3)+C(s-1)=s^2-9s+22$$

$$\left.\begin{aligned} A+B&=1\\ 6A+2B+C&=9\\ 9A-3B-C&=22 \end{aligned}\right\} A=2,\ B=-1, C=-1$$

$$\mathcal{L}^{-1}\left\{\frac{s^2+9s+22}{(s-1)(s+3)^2}\right\}=\mathcal{L}^{-1}\left\{\frac{2}{s-1}-\frac{1}{s+3}-\frac{1}{(s+3)^2}\right\}$$

$$=2e^t-e^{-3t}-te^{-3t}$$

By inversion of first shift theorem,

$$a=3,\ F(s+3)=\frac{1}{(s+3)^2}\Rightarrow F(s)=\frac{1}{s^2}\Rightarrow f(t)=t.\ \text{Hence, } \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\}=te^{-3t}$$

3.4 CONVOLUTION THEOREM

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{G(s)\} = g(t)$ then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-u)g(u)\,du$$

$$F(s) \xrightarrow{\mathcal{L}^{-1}\{\ \}} f(t) \longrightarrow f(t-u)$$

$$G(s) \xrightarrow{\mathcal{L}^{-1}\{\ \}} g(t) \longrightarrow g(u)$$

Example: Use convolution theorem to find the Laplace inverse of the following functions.

(a) $$\frac{4}{s^2 - 9}$$

(b) $$\frac{2}{s^2\left(s^2 + 1\right)}$$

Solution:

(a)

$$\frac{4}{s^2-9}=\frac{4}{(s-3)(s+3)}$$

$$F(s)=\frac{4}{(s-3)} \quad , \quad G(s)=\frac{1}{(s+3)}$$

$$f(t)=4e^{3t} \quad , \quad g(t)=e^{-3t}$$

$$f(t-u)=4e^{3t}e^{-3u} \quad , \quad g(u)=e^{-3u}$$

$$\mathcal{L}^{-1}\left\{\frac{4}{s^2-9}\right\}=\int_0^t 4e^{3t}e^{-3u}e^{-3u}\,du=4e^{3t}\int_0^t e^{-6u}\,du=4e^{3t}\left[\frac{e^{-6u}}{-6}\right]_0^t$$

$$=4e^{3t}\left[\frac{e^{-6t}}{-6}+\frac{1}{6}\right]=-\frac{2}{3}e^{-3t}+\frac{2}{3}e^{3t}$$

(b)

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2(s^2+1)}\right\}=\mathcal{L}^{-1}\{F(s)G(s)\}$$

$$F(s)=\frac{2}{s^2},\quad G(s)=\frac{1}{(s^2+1)}$$

$$f(t)=2t,\qquad g(t)=\sin t$$

$$f(t-u)=2(t-u),\qquad g(u)=\sin u$$

$$\int_0^t 2(t-u)\sin u\,du=2\left(-(t-u)\cos u-\sin u\right)\Big|_0^t$$

$$=2(-\sin t)-2(-t\cos 0-\sin 0)$$

$$=-2\sin t+2t$$

Tabular Method

	Diff	*Int*
$(+)$	$t-u$	$\sin u$
$(-)$	-1	$-\cos u$
$(+)$	0	$-\sin u$

3.5 LAPLACE TRANSFORM OF INTEGRAL

If $\mathcal{L}\{f(t)\} = F(s)$

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$$

$$f(u) \longrightarrow f(t) \xrightarrow{\mathcal{L}\{\ \}} F(s)$$

Example: Find the Laplace transform of the integrals.

(a) $$\int_0^t 3u + 7\, du$$

(b) $$\int_0^t u\cos 2u \; du$$

Solution:

(a)

$$f(u) = 3u + 7$$

$$f(t) = 3t + 7$$

$$F(s) = \mathcal{L}\{3t+7\} = \frac{3}{s^2} + \frac{7}{s}$$

$$\mathcal{L}\left\{\int_0^t 3u + 7\,du\right\} = \frac{1}{s}\left(\frac{3}{s^2} + \frac{7}{s}\right)$$

$$= \frac{3}{s^3} + \frac{7}{s^2}$$

Solution:

(a)

$$f(u) = u\cos 2u$$

$$f(t) = t\cos 2t$$

$$F(s) = \mathcal{L}\{t\cos 2t\}$$

$$= \frac{s^2-4}{\left(s^2+4\right)^2}$$

$$\mathcal{L}\left\{\int_0^t u\cos 2u\ du\right\} = \frac{s^2-4}{s\left(s^2+4\right)^2}$$

Using derivative of t-transform

$$\mathcal{L}\{t\cos 2t\} = (-1)^1\frac{dF(s)}{ds}$$

$$n=1,\quad f(t)=\cos 2t$$

$$F(s) = \frac{s}{s^2+4}$$

$$\mathcal{L}\{t\cos 2t\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= (-1)\left[\frac{\left(s^2+4\right)-2s^2}{\left(s^2+4\right)^2}\right]$$

$$= \frac{s^2-4}{\left(s^2+4\right)^2}$$

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