DISCRETE MATHEMATICS AND APPLICATIONS

Sets

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Chapter Description

- Chapter outline
  - 2.1 Set terminologies and concepts
  - 2.2 Operation on sets
  - 2.3 Cartesian products
  - 2.4 Power sets
  - 2.5 Applications of set theory
- Aims
  - Write and define a set in different notation
  - Identify the element of a set, empty set, set equality, subset and cardinality of a set
  - Use set operation and identities to solve problem in set theory
  - Identify the Cartesian product of two or more sets
  - Identify the power set of a given set and its number of elements
  - Apply the knowledge of set theory into real world problem

References

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Sets and Empty Set (i)

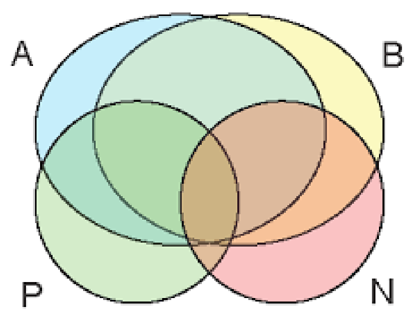
Definition: A **set** is an unordered collection of elements.

Let A be a set, and $a_1, a_2, ...a_n$ is the elements, then $A = \{a_1, a_2 ... a_n\}$.

If a_1 is an element of A **OR** a_1 belongs to A, then $a_1 \in A$.

If a_1 is not an element of A, then $a_1 \notin A$.

Definition: A set with no elements is known as **empty set**, denoted by $\varnothing$ or $\{ \ \}$



Sets and Empty Set (ii)

Example 1

A set of local cars can be expressed as, *Cars* = {Preve, MyVi, Kancil, Saga, Waja}

A set of food and beverages in a restaurant menu, *F* ={Roti canai, teh ais, mi goreng, air suam}

A set of alphabets can be written as, *A*= {a, b, c, …, z}

Example 2

$\mathbf{N}$ = the set of natural numbers

$\mathbf{Q}$ = the set of rational numbers

$\mathbf{R}$ = the set of real numbers

$\mathbf{P}$ = the set of prime numbers

$\mathbf{Z}$ = the set of integers

$\mathbf{E}$ = the set of even integers

$\mathbf{O}$ = the set of odd integers

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Set Notation (i)

There are two types of set notations which are roster and set builder:

1. Roster

List the elements of a set in the form of

Set_name={element_1,element_2,…element_n}

i.e $N = \{1, 2, 3, 4, 5, \ldots n\}$

2. Set Builder

Set_name={variable|$\text{variable\_condition}_1$, $\text{variable\_condition}_2$,…$\text{variable\_condition}_n$}

i.e $N = \{x \mid x \text{ is natural number}\}$

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Set Notation (ii)

Example 1

i. By using roster notation, express the set of positive integer less than 5.

Answer: $A = \{1, 2, 3, 4\}$

ii. By using set builder notation, express the set of positive integer less than 5.

Answer: $A = \{x \mid x < 5, x \in Z^+\}$

Example 2

i. By using roster notation, express a set of even number between 11 until 21.

Answer: $A = \{12, 14, 16, 18, 20\}$

ii. By using set builder notation, express a set of even number between 11 until 21.

Answer: $A = \{x \mid 11 < x < 21, x \text{ is even}\}$ **OR**

$A = \{x \mid 11 < x < 21, x \in Z^+, x = 2n \text{ where } n = 1, 2, \ldots n\}$

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Set Notation (iii)

Example 3

i. By using roster notation, express a set of number where the element, $x = \frac{n^2}{2}, n = 1, 2, \ldots 5$.

Answer: $A = \{\frac{1}{2}, 2, \frac{9}{2}, 8, \frac{25}{2}\}$ or $A = \{0.5, 2, 4.5, 8, 12.5\}$

ii. By using set builder notation, express a set of number where the element, $x = \frac{n^2}{2}, n = 1, 2, \ldots 5$.

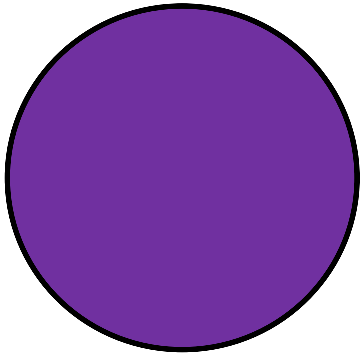
Answer: $A = \{x \mid x = \frac{n^2}{2}, n = 1, 2, \ldots, 5\}$

Venn diagram

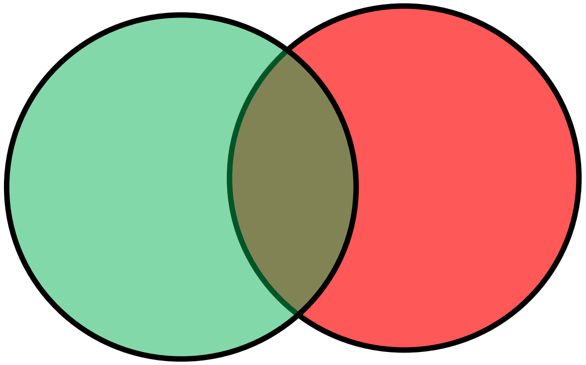
Definition: A **Venn diagram** is a diagram that shows all possible logical relations between a finite collection of different sets.

Venn diagram is usually represented by overlapping circles which are shaded according to the characteristics or relationship of the set(s).

Set A



Set A and Set B



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Equal sets (i)

Definition: Two sets A and B**,** are **equal sets** $A = B$, if and only if they have the same elements regardless of the order and repetitive elements.

Example 1

Given set $X = \{a, b, c\}$, $Y = \{b, a, c\}$, $Z = \{a, a, b, b, c, c, c\}$. Determine whether set X, Y and Z are equal sets.

Answer:

$X = Y$

$X = Z$

$X = Y = Z$

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Equal sets (ii)

Example 2

Set $A = \{1, 3, 5\}$ and $B = \{1, 3, 1\}$ are equal. Is this statement true? Justify your answer

Answer:

A and B is not equal, $A \neq B$ as 5 is not an element in B. Thus, they don't have the same elements.

Example 3

Suppose there are two sets which are a set of vowels and consonants. Are these two sets equal?

Answer:

It is not equal as these two sets clearly have different elements.

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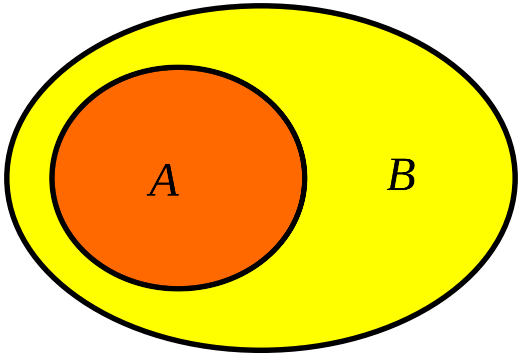
Subset (i)

Definition: A set A is a **subset** of a set B, $A \subseteq B$, if and only if every element of A is an element of B.

For any set

i. $\varnothing \subseteq B$
ii. $B \subseteq B$



If a set A is not a subset of a set B, $A \not\subseteq B$.

Definition: A set A is a **proper subset** of a set B, $A \subset B$, if A is a subset of B and there exist one element of B that is not in A, where $A \neq B$

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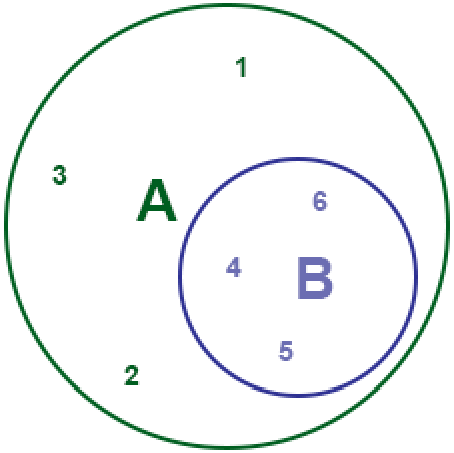
Subset (ii)

The definition of subset and proper subset is **not the same**.

A subset contains all elements which exist in the original set.



B is subset of A

Proper subset is a set which is

1. Definitely "*Smaller*" with fewer element than a '*bigger*' set and
2. All elements exist in a bigger set.

In the example, $A \subseteq A$ and $B \subseteq A$. However, we can also write $B \subset A$ as it has less elements than *A* with all elements exist in *A*. The representation of $B \subset A$ is more appropriate mathematically.

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Subset (iii)

Example 1

Let $A = \{2, 4\}$ and $B = \{1, 2, 3, 4, 5\}$

Then $A \subseteq B$ or $A \subset B$, and $B \not\subset A$

Example 2

Let $Set1 = \{Monday, Tuesday\}$, $Set2 = \{Tuesday, Thursday, Friday\}$ and $Set3$ is a set of days in a week.

Then $Set1 \not\subset Set2$ and $Set1 \subseteq Set3$

$Set2 \not\subset Set1$ and $Set2 \subset Set3$

$Set3 \not\subset Set1$ and $Set3 \not\subset Set2$

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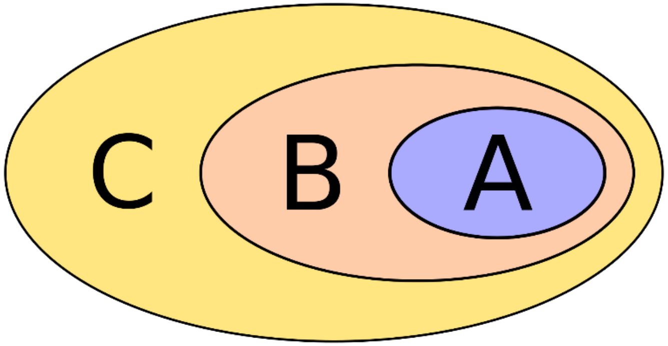
Subset (iv)

Example 3

If $A \subseteq B$ and $B \subseteq C$, show that $A \subseteq C$



Answer:

Suppose an element, a exist in Set A where $a \in A$
Let $A \subseteq B$, then $a \in B$ where a is an element of B
If $B \subseteq C$, then $a \in C$ as $a \in B$
Thus, $A \subseteq C$ $\therefore$ proven.

Relate it with CHAPTER 4: Proof Methods !!!

Finite & Infinite Sets

Definition: Finite set is a set that contains finite elements.

Definition: Infinite set is a set that contains infinite elements.

Example 1

Let O is a set of natural number less than 10. Then O is finite. $O = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Example 2

If O is a set of natural number. Then O is infinite. $O = \{1, 2, 3, \ldots\}$

Example 3

Let set $A \subseteq B$. Determine whether A is finite or infinite, given that B is finite.

If $A \subseteq B$, then $a_n \in A$ and $a_n \in B$

If B is finite, then $\exists a_{n-1} \le a_n$ where $a_{n-1} \in B$

Thus $a_{n-1} \in A$. As $a_{n-1} \le a_n$ $\therefore$ A is finite.

Relate it with CHAPTER 4: Proof Methods !!!

Cardinality of Set (i)

Definition: If A is finite, then the **cardinality of set**, $|A|$ is the number of distinct elements in A.

Example 1

Let V is a set of vowels.

$V = \{a, e, i, o, u\}$

$|V| = 5$, the cardinality of V is 5.

Example 2

Empty set, $\varnothing$ has no elements.

$|\varnothing| = 0$

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Cardinality of Set (ii)

Example 3

Let $H = \{5, 3, 1, 3, 1, 3, 4\}$

$|H| = 4$ as there are only 4 distinct elements, $1, 3, 4, 5 \in H$

Example 4

Let $Z^+ = \{1, 2, 3, \ldots, n, n+1, n+2\}$

$|Z| = n + 2$

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Operations on Sets

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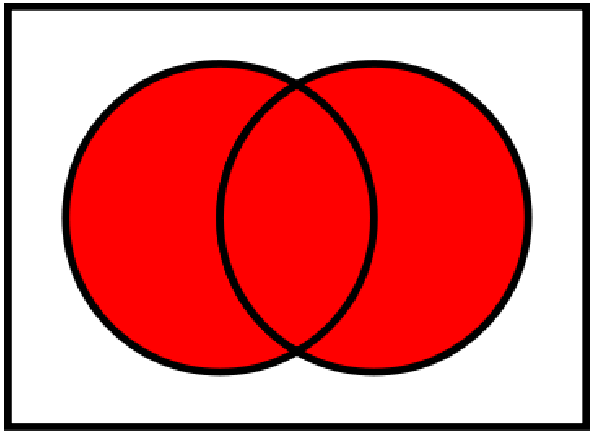
Union of Sets (i)

Definition: The **union of sets** A and B, $A \cup B$ is the set which contains all elements that are belong **either** in A or B.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$



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Union of Sets (ii)

Example 1

Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$.

$A \cup B = \{1, 2, 3, 4, 5\}$

*Repetition of elements is not allowed as it counts as the same element in $A \cup B$.

Example 2

Let $A = \{i, a, 8\}$ and $B = \{7, 8, 9\}$

$A \cup B = \{a, i, 7, 8, 9\}$

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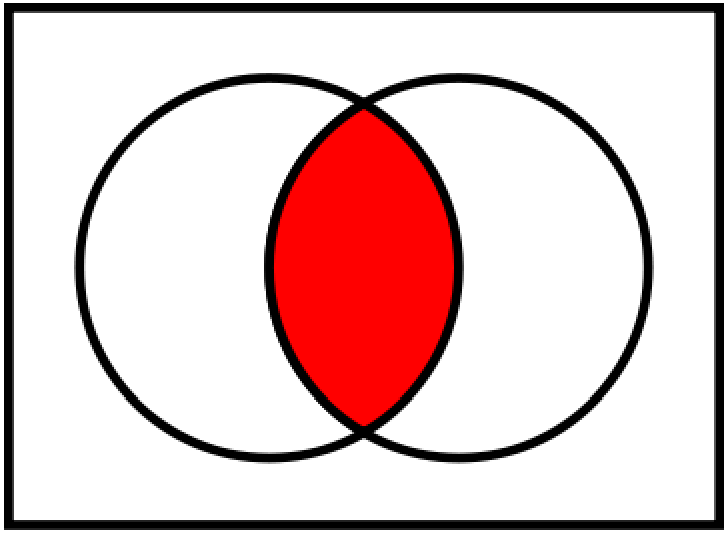
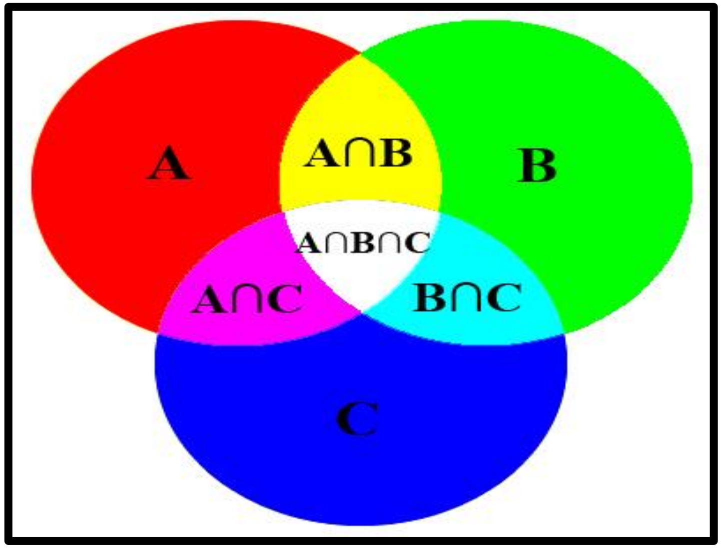
Intersection of sets (i)

Definition: The **intersection of sets** A and B, $A \cap B$ is the set which contains **common** elements of both A and B.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$





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Intersection of sets (ii)

Example 1

Let $X = \{1, 2, 3\}$ and $Y = \{3, 4, 5\}$.

$X \cap Y = \{3\}$

* Repetition of elements is not needed.

Example 2

Let $A = \{a, e, i, o, u\}$ and $B = \{i, i, o, a, n\}$. Determine $|A \cap B|$.

Then $A \cap B = \{a, i, o\}$.

Thus $|A \cap B| = 3$

Collection of sets (i)

Definition: Union collection of sets is the set that contain all elements that are belong either in sets $A_1, A_2, ..., A_n,$

$$\bigcup_{i=1}^{n} A_i = A_1, A_2, ..., A_n,$$

Definition: Intersection Collection of Sets is the set that contain common elements in sets $A_1, A_2, ..., A_n,$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap ... \cap A_n$$

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Collection of sets (ii)

Example 1

Let $A_i = \{1, 2, 3, \ldots i\}$. Find $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$

Answer

Determine $A_1, A_2, A_3 \ldots A_n$ and find $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$

$$A_1 = \{1\}$$
$$A_2 = \{1, 2\}$$
$$A_3 = \{1, 2, 3\}$$
$$A_1 \cup A_2 \cup A_3 = \{1, 2, 3\}$$
$$A_1 \cap A_2 \cap A_3 = \{1\}$$

If we extrapolate the idea, we will see that

$\bigcup_{i=1}^{n} A_i = \{1, 2, 3, \ldots, i\}$ and $\bigcap_{i=1}^{n} A_i = \{1\}$

Collection of sets (iii)

Example 2

Let $A_i = \{0, i\}$. Find $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$

Answer

Determine $A_1, A_2, A_3 \ldots A_n$ and find $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$

$$A_1 = \{0,1\}$$
$$A_2 = \{0,2\}$$
$$A_3 = \{0,3\}$$
$$A_1 \cup A_2 \cup A_3 = \{0,1,2,3\}$$
$$A_1 \cap A_2 \cap A_3 = \{0\}$$

If we extrapolate the idea, we will see that

$\bigcup_{i=1}^{n} A_i = \{0,1,2,3,\ldots,i\}$ and $\bigcap_{i=1}^{n} A_i = \{0\}$

Collection of sets (iv)

Example 3

Let $A_i = \{i, i+1, i+2, \ldots\}$. Find $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$

Answer

Determine $A_1, A_2, A_3 \ldots A_n$ and find $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$

$$A_1 = \{1, 2, 3, \ldots\}$$
$$A_2 = \{2, 3, 4, \ldots\}$$
$$A_3 = \{3, 4, 5, \ldots\}$$
$$A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, \ldots\}$$
$$\bigcap_{i=1}^{2} A_2 = A_1 \cap A_2 = \{2, 3, 4, \ldots\}$$
$$\bigcap_{i=1}^{3} A_3 = A_1 \cap A_2 \cap A_3 = \{3, 4, 5, \ldots\}$$

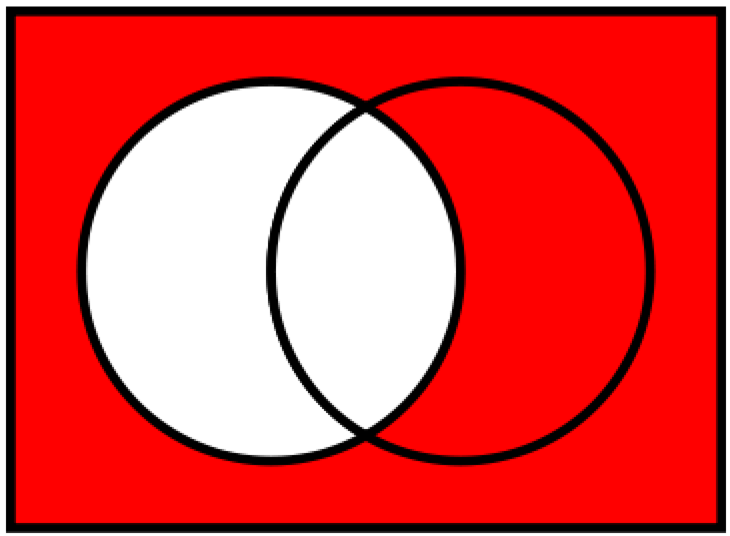
If we extrapolate the idea, we will see that

$\bigcup_{i=1}^{n} A_i = \{0, 1, 2, 3, \ldots, i\}$ and $\bigcap_{i=1}^{n} A_i = \{n, n+1, n+2, \ldots\}$

Complement of set (i)

Definition: Let A be the universal set. The **complement** of set A, $\overline{A}$ or A^C is the set which contains elements in U but not in A.

$$\overline{A} = \{x \mid x \in U, x \notin A\}$$



Complement of set (ii)

Example 1

Let U is a set of English alphabets and A is a set of consonants. Then $\overline{A} = \{a, e, i, o, u\}$

Example 2

Let set $U = \{1, 2, 3, ...10\}$ and E is set of odd number less than 10. Find $\overline{E}$.

Answer:

As $E = \{1, 3, 5, 7, 9\}$, then $\overline{E} = \{2, 4, 6, 8, 10\}$

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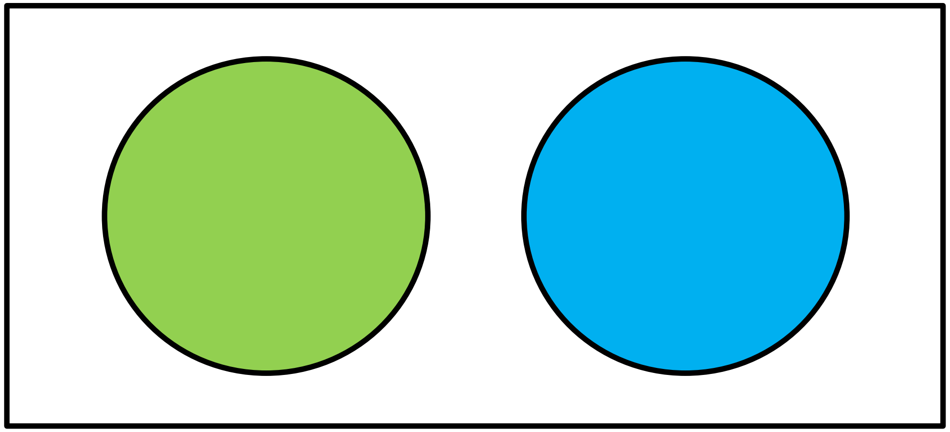
Disjoint set

Definition: Set A and set B are **disjoint**, if they have no common elements, $A \cap B = \varnothing$.



Example 1

i. A set of consonants and a set of vowels are disjoint.

ii. A set of boys and girls are disjoint.

iii. Set $A = \{1,3,5,7\}$ and set $B = \{2,4,9\}$ are disjoint.

Laws of sets

Let U be a universal set. If A, B, and C are subsets of U, then,

Laws	Properties
Identity	$A \cup \varnothing = A$ $A \cap U = A$
Idempotent	$A \cup A = A$ $A \cap A = A$
Commutative	$A \cup B = B \cup A$ $A \cap A = B \cap A$
Associative	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$

Laws	Properties
Distributive	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Absorption	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Complement	$\overline{\overline{A}} = A$ $\overline{U} = \varnothing$ $\quad \overline{\varnothing} = U$ $A \cup \overline{A} = U$ $\quad A \cap \overline{A} = \varnothing$
De Morgan	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$

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Addition of sets (i)

Let *A*, *B*, and *C* be finite sets. The addition formulae for A, *B*, and *C* depends on these conditions:

1. If *A* and *B* are disjoint

$$|A \cup B| = |A| + |B|$$

2. If *A* and *B* are not disjoint

$$|A \cup B| = |A| + |B| - |A \cap B|$$

3. If *A*, *B*, and *C* are not disjoint

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Addition of sets (ii)

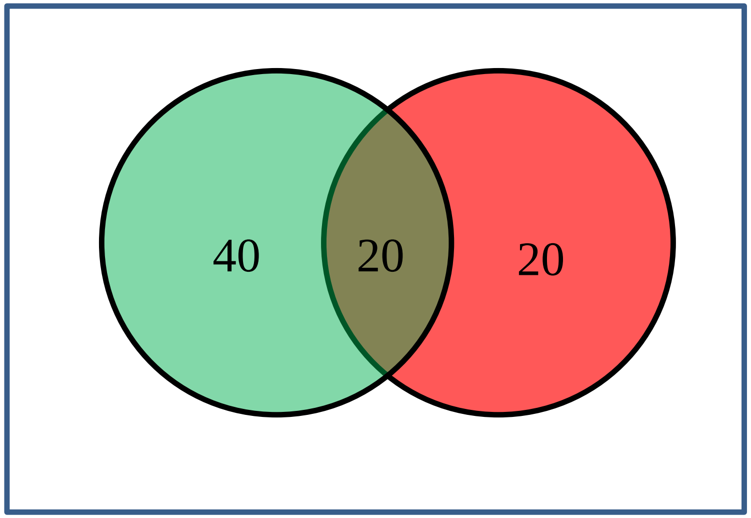
Let A and B be subsets of universal set U where $|U|=100, |A|=60, |B|=40. |A \cap B|=20.$

a) $|A \cup B| = |A| + |B| - |A \cap B|$

$$= 60 + 40 - 20$$
$$= 80$$

b) $|\bar{A}| = |U - A| = 100 - 60 = 40$

$|\bar{B}| = |U - B| = 100 - 40 = 60$



c) $|A \cap \bar{B}| = |A| - |A \cap B|$

$$= 60 - 20$$
$$= 40$$

d) $|A \cup \bar{B}| = |A| + (|U| - |B|)$

$= |A|$

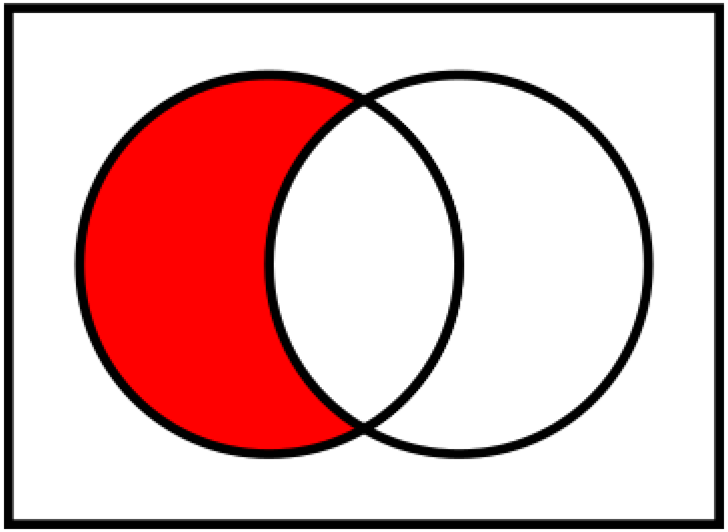
0

It is better to sketch Venn diagram before answering or use numbering system.

Difference of sets

Definition: Let A, and B be finite sets. The **difference of sets** A and B, $A-B$, is the set which contains all elements in A but not B.

$$A-B=\{x \mid x\in A, x\notin B\}$$



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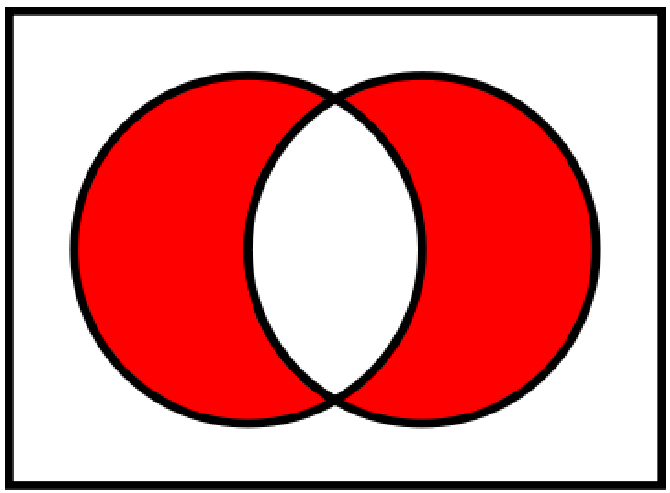
Symmetric difference/ Mutually exclusive

Definition: Let A, and B be finite sets. The **symmetric difference/ mutually exclusive** of A and B, $A \oplus B$, is the set which contains elements either in A or B but not both.

$$A \oplus B = \{x \mid (x \in A \cup \in B) \wedge (x \notin A \cap B)\}$$

$$A \oplus B = (A - B) \cup (B - A)$$



Cartesian Products (i)

Definition: A general set may be unordered. An ordered set which has sequence of $(a_1, a_2, ..., a_n)$ is known as **ordered *n*-tuple.**

A 2-ordered *n*-tuple are equal if and only if

$(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n)$

$(a_i) = (b_i), \forall i = 1, 2, ..., n$

Ordered pairs between element of (a_i) and (b_i) exist $\forall i = 1, 2, ..., n$.

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Cartesian Products (ii)

Definition: The **Cartesian product** of set A and B, $A \times B$ is the set of all ordered pairs $(a_i, b_i), \forall i = 1, 2, ..., n$ such that $a_i \in A$ and $b_i \in B$.

The **cardinality of Cartesian product** is given as $|A \times B| = |A| \times |B|$.

Suppose there are n sets. The Cartesian product of set $A_1, A_2, ..., A_n$, $A_1 \times A_2 \times ... \times A_n$ is the set of all ordered n-tuples of $\{(a_1, a_2, ..., a_n)\}, \forall i = 1, 2, ..., n$ such that $a_i \in A$.

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Power sets (i)

Definition: The **power set** of set A, $P(A)$ is the set of all subsets of A. The number of distinct subset of a set A with n elements is 2^n.

Example 1

Let $A = \{1, 2\}$, as A has 2 elements, then the number of distinct subset of A is $2^2 = 4$.

$P(A)$={ $\varnothing$, {1}, {2}, {1,2}}.

*Don't forget to include empty set as $\varnothing \in A$ for any sets.

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Power sets (ii)

Example 2

Determine the power set of $\varnothing$.

As $\varnothing$ has no elements, then $2^0 = 1$.

$P(\varnothing) = \{\varnothing\}$ as $\varnothing \in A$

Example 2

Determine the power set of $A = \{\varnothing, \{\varnothing\}\}$. There are 2 elements, so $2^2 = 4$

$P(A) = \{\varnothing, \varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}$

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