

DISCRETE MATHEMATICS AND APPLICATIONS

Number Theory 3

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Chapter Description

- Chapter outline
 - 1.6 Euclidean Algorithm
 - 1.7 Extended Euclidean Algorithm
 - 1.8 Modular Arithmetic
- Aims
 - Find the Greatest Common Divisor of two integers by using Euclidean Algorithm
 - Find the linear equation between two numbers and their Greatest Common Divisor



Solve modular arithmetic problems

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References

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Euclidean algorithm

- Euclidean algorithm is another method to determine GCD
- This method is efficient than prime factorization especially if the given integers are large
- The algorithm steps is given as follows
 - Step 1: Initialize. Let two integers a and b
 - Step 2: If $a > b$, then use division algorithm to determine $b = qa + r$. Else $a = qb + r$
 - Step 3: q will becomes new dividend and r becomes new divisor
 - Step 4: Repeat Step 2 until $r = 0$
 - Step 5: The last divisor is the $\text{GCD}(a, b)$



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Euclidean algorithm: Example

Determine $\text{GCD}(190,34)$ by using Euclidean algorithm.

Let $a = 190$ and $b = 34$. As $34 < 190$ then,

divide 190 by 34,	$190 = 5(34) + 20$
divide 34 by 20,	$34 = 1(20) + 14$
divide 20 by 14,	$20 = 1(14) + 6$
divide 14 by 6,	$14 = 2(6) + 2$
divide 6 by 2,	$6 = 3(2) + 0, r=0 \text{ STOP}$

2 is the last divisor



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Extended Euclidean Algorithm

Theorem

Let a and b be positive integers, then there exist integers s and t such that

$$\text{GCD}(a,b)=sa+tb$$

The theorem states that the GCD for a and b can be expressed as a linear combination of a and b



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Extended Euclidean Algorithm: Example

Express $\gcd(252, 198) = 18$ as a linear combination of 252 and 198.

Use Euclidean algorithm first to produce these linear equations

$$252 = 1(198) + 54$$

$$198 = 3(54) + 36$$

$$54 = 1(36) + 18$$

$$36 = 2(18)$$

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Extended Euclidean Algorithm: Example

$$54 = 252 - 1(198)$$

$$\begin{aligned} 36 &= 198 - 3(54) \\ &= 198 - 3[252 - 1(198)] \\ &= 198 - 3(252) + 3(198) \\ &= 4(198) - 3(252) \end{aligned}$$

$$\begin{aligned} 18 &= 54 - 1(36) \\ &= 252 - 1(198) - 1[4(198) - 3(252)] \\ &= 252 - 198 - 4(198) + 3(252) \\ &= 4(252) - 5(198) \end{aligned}$$



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Modular arithmetic

In some real life situation which involves repeated trend or cycle of a process, we can represent it by using modular arithmetic. Modular arithmetic only concern on the calculation of the remainder only. For example:

If the time is now 9 o'clock, what time will it be 100 hours from now?

Let $9 + 100 = 109$ and we use 24 hour system. Therefore the divisor will be 24

$$109 = 4(24) + 13$$

The remainder is 13

In 100 hours it will be 1300 or 1 p.m



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Modular arithmetic: Example

a) $17 \bmod 3$

$$17 = 5(3) + 2$$

$$r = 2$$

b) $133 \bmod 9$

$$-133 = -15(9) + 2$$

$$r = 2$$

c) $2004 \bmod 101$

$$2004 = 19(101) + 85$$

$$r = 85$$

d) $29 \bmod 5$

$$29 = 5(5) + 4$$

$$r = 4$$

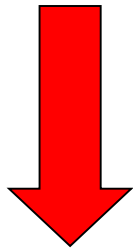


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Congruence

$$m = qn + r$$



$$m \equiv r \pmod{n}$$

$m \not\equiv r \pmod{n}$
m and r **not congruent** modulo n

$$m \pmod{n} = r \pmod{n}$$

congruent to r modulo n

$$n \mid (m - r)$$

n is modulus

Example

$$\text{a) } 29 = 5(5) + 4 \Rightarrow 29 \equiv 4 \pmod{5}$$

$$\text{b) } 3 = 0(6) + 3 \Rightarrow 3 \equiv 3 \pmod{6}$$



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Mod-n Function

For each $n \in \mathbb{Z}^+$, we define a function f_n , the mod- n function, as follows:

If z is a nonnegative integer, then

$$f_n(z) = r, \text{ with } z = r(\text{mod } n) \text{ and } 0 \leq r < n.$$

Example:

$$f_3(16) = 1 \text{ because } 16 = 5(3) + 1 \text{ and } 16 \equiv 1(\text{mod } 3)$$

$$f_7(156) = 2 \text{ because } 156 = 22(7) + 2 \text{ and } 156 \equiv 2(\text{mod } 7)$$

$$f_3(14) = 2 \text{ because } 14 = 4(3) + 2 \text{ and } 14 \equiv 2(\text{mod } 3)$$

$$f_7(153) = 6 \text{ ??????}$$



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- If f is the **mod-7 function**, solve $f(z) = 2$.

Solution:

$$f_7(z) = 2 \quad \Leftrightarrow \quad z \equiv 2 \pmod{7}$$

$$\Leftrightarrow \quad z = q(7) + 2$$

$$\text{if } q = 0; \quad z = 0(7) + 2 = 2$$

$$\text{if } q = 1; \quad z = 1(7) + 2 = 9$$

$$\text{if } q = 2; \quad z = 2(7) + 2 = 16$$

$$\text{if } q = 3; \quad z = 3(7) + 2 = 23$$

Therefore, the solution of $f(z) = 2$ is $\{2, 9, 16, 23, \dots\}$



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