BMM3553 Mechanical Vibrations

Chapter 3: Damped Vibration of Single Degree of Freedom System (Part 2)

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Chapter Description

- Expected Outcomes

  Students will be able to:

  - Determine the Response of damped system under harmonic motion
  - Solve the problem related to damped SDOF Force vibration

- References
  - Singiresu S. Rao. Mechanical Vibrations. 5th Ed
  - Abdul Ghaffar Abdul Rahman. BMM3553 Mechanical Vibration Note. UMP.
  - Md Mustafizur Rahman. BMM3553 Mechanical Vibration Lecture Note. UMP

Response of a Damped System Under Harmonic Force

The general equation of motion for SDOF damped system is

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos\omega t$$

Let, the particular solution $x_p(t) = X\cos(\omega t - \phi)$

$$\therefore \dot{x}_p(t) = -X\omega\sin(\omega t - \phi)$$

and $\ddot{x}_p(t) = -X\omega^2\cos(\omega t - \phi)$

$$-mX\omega^2\cos(\omega t - \phi) - cX\omega\sin(\omega t - \phi) + kX\cos(\omega t - \phi) = F_0\cos\omega t$$

Response of a Damped System Under Harmonic Force

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The equation of motion

$$X\left[\left(k-m\omega^2\right)\cos(\omega t-\phi)-c\omega\sin(\omega t-\phi)\right]=F_0\cos\omega t$$

$$X\left[\left(k-m\omega^2\right)\cos\phi+c\omega\sin\phi\right]=F_0$$

$$X\left[\left(k-m\omega^2\right)\sin\phi-c\omega\cos\phi\right]=0$$

The solution gives

$$X=\frac{F_0}{\left[\left(k-m\omega^2\right)^2+c^2\omega^2\right]^{1/2}}$$

and

$$\phi=\tan^{-1}\left(\frac{c\omega}{k-m\omega^2}\right)$$

Response of a Damped System Under Harmonic Force

Substituting the following,

$$\omega_n = \sqrt{\frac{k}{m}}; \quad \delta_{st} = \frac{F_0}{k}; \quad \frac{c}{m} = 2\zeta\omega_n; \quad r = \frac{\omega}{\omega_n}$$

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

and

$$\phi = \tan^{-1}\left\{\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right\} = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

Response of a Damped System Under Harmonic Force

We obtain

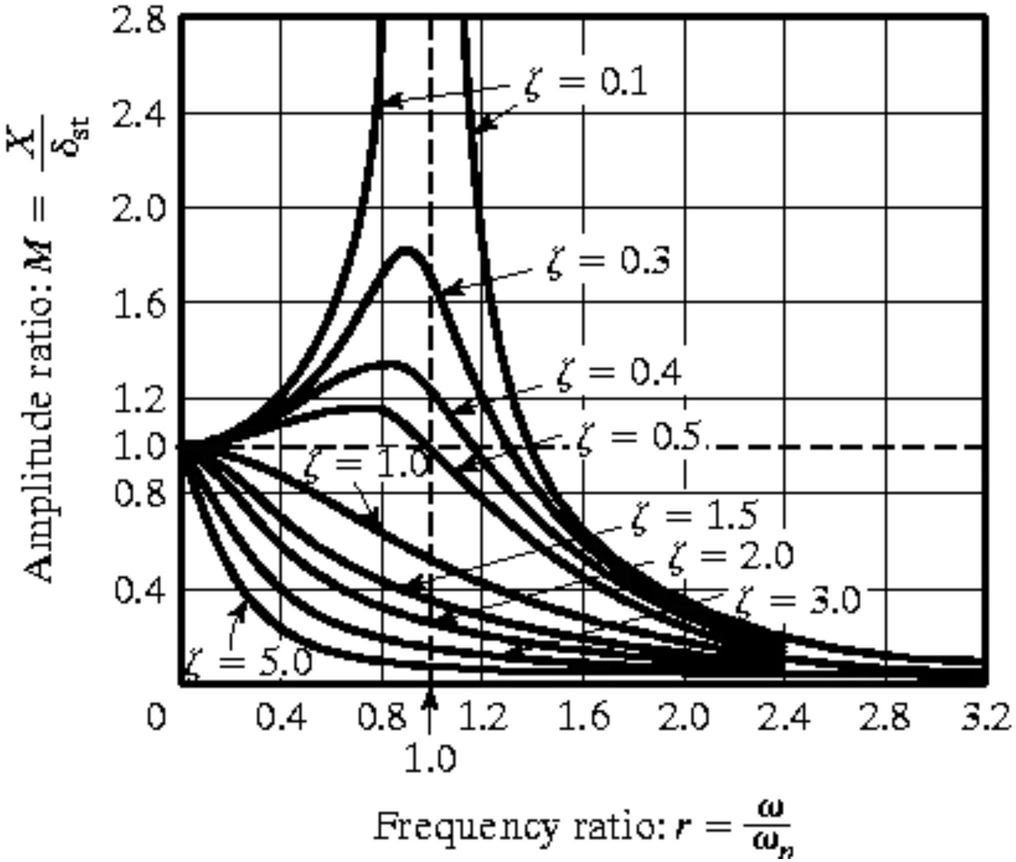
$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

and

$$\phi = \tan^{-1}\left\{\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right\} = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

Response of a Damped System Under Harmonic Force

The magnification factor (M) for different damping ratio and frequency ratio is shown in figure below:



$$M = \frac{X}{\delta_{st}},$$

$$r = \frac{\omega}{\omega_n}$$

$$\zeta = \frac{c}{C_c}$$

Image source: S.S. Rao 5th Ed.

Response of a Damped System Under Harmonic Force

The characteristics of phase angle due to the effect of damping

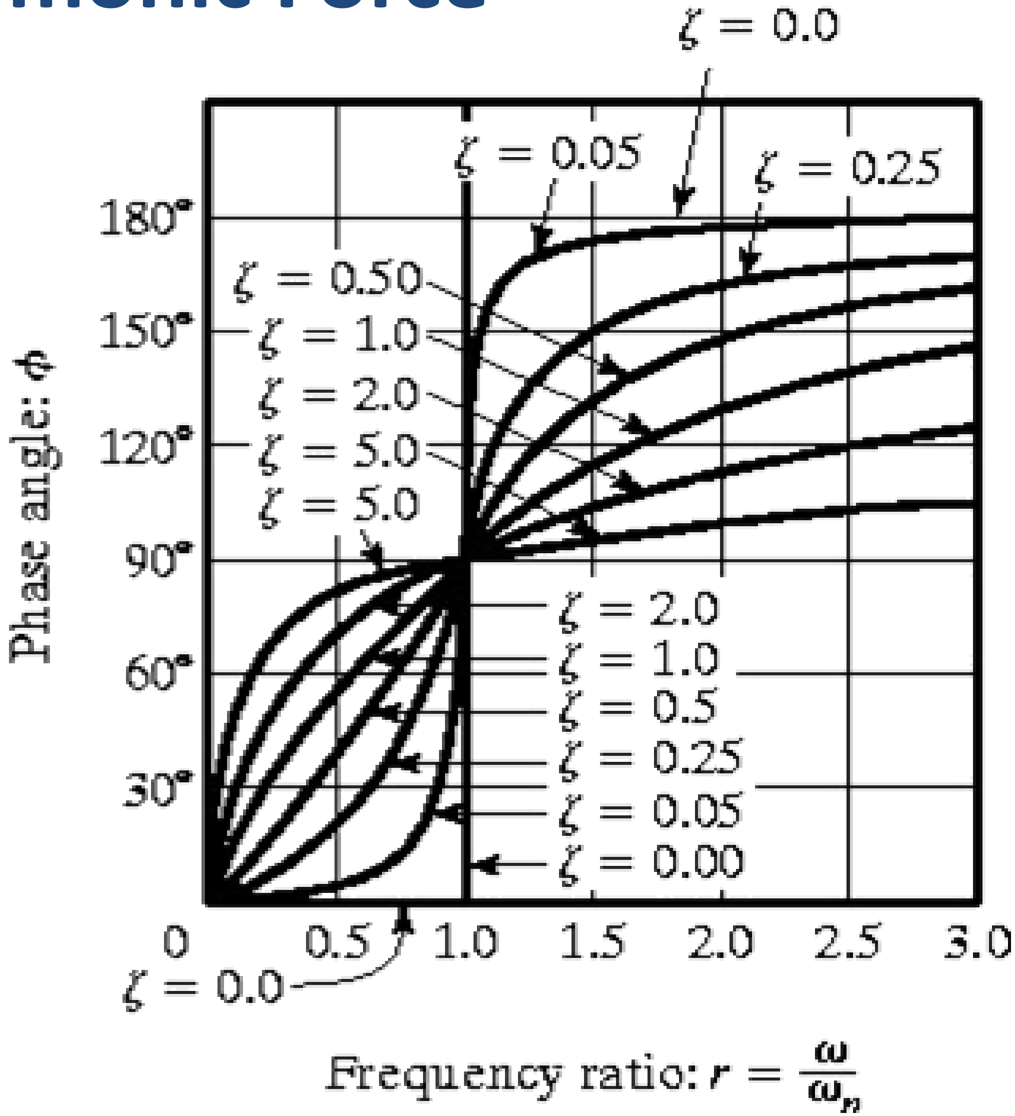


Image source: S.S. Rao 5th Ed.

Total Response

The total response is $x(t) = x_h(t) + x_p(t)$

$$x_h(t) = X_0 e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_0)$$

where, $\omega_d = \omega_n \sqrt{1-\zeta^2}$

$$x_p(t) = X \cos(\omega t - \phi)$$

The complete solution is

$$x(t) = X_0 e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_0) + X \cos(\omega t - \phi)$$

Total Response

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$$\phi = \tan^{-1}\left\{\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right\} = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

For the initial conditions

$$t=0, \quad x(0)=x_0 \text{ and } \dot{x}(0)=\dot{x}_0$$

$$x(t)=X_0 e^{-\zeta\omega_n t}\cos(\omega_d t-\phi_0)+X\cos(\omega t-\phi)$$

$$x_0 = X_0\cos\phi_0 + X\cos\phi$$

$$\dot{x}_0 = -\zeta\omega_n X_0\cos\phi_0 + \omega_d X_0\sin\phi_0 + \omega X\sin\phi$$

Exercise

Example 3.2 (S.S. Rao 5th Ed)

Find the total response of a single degree of freedom system with m = 10 kg, c = 20 N-s/m, k = 4000 N/m, x_0 = 0.01 m, $\dot{x}_0 = 0$ under the following conditions:

a. An external force $F(t) = F_0 \cos \omega t$ acts on the system with $F_0 = 100$ N and $\omega = 10$ rad/s.

b. Free vibration with $F(t) = 0$.

Solution

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{20}{2\sqrt{(4000)(10)}} = 0.05$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-(0.05)^2}\,(20) = 19.974984 \text{ rad/s}$$

$$\delta_{\text{st}} = \frac{F_0}{k} = \frac{100}{4000} = 0.025\text{m}$$

$$r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5$$

$$X = \frac{\delta_{st}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.025}{\left[(1-0.05^2)^2 + (2\cdot 0.5\cdot 0.5)^2\right]^{1/2}} = 0.03326\text{m}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2\cdot 0.05\cdot 0.5}{1-0.5^2}\right) = 3.814075^\circ$$

Using initial conditions $x_0 = 0.01$ and $\dot{x}_0 = 0$

$$x_0 = X_0 \cos\phi_0 + X\cos\phi$$

$$0.1 = X_0\cos\phi_0 + (0.03326)(0.997785)$$

$$X_0\cos\phi_0 = -0.023186$$

$$\dot{x}_0 = -\zeta\omega_n X_0\cos\phi_0 + \omega_d X_0 \sin\phi_0 + \omega X \sin\phi$$

$$0 = -(0.05)(20)X_0\cos\phi_0 + X_0(19.974984)\sin\phi + (0.03326)(10)\sin(3.814075^\circ)$$

$$X_0\sin\phi_0 = -0.002268$$

Solution

Hence,

$$X_0 = \left[(X_0 \cos\phi_0)^2 + (X_0 \sin\phi_0)^2 \right]^{1/2} = 0.023297$$

and $$\tan\phi_0 = \frac{X_0 \sin\phi_0}{X_0 \cos\phi_0} = 0.0978176$$

$$\boxed{\phi_0 = 5.586765^\circ}$$

b. For free vibration, the total response is

$$x(t) = X_0 e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_0)$$

Using the initial conditions,

$$X_0 = \left[x_0^2 + \left(\frac{\zeta\omega_n x_0}{\omega_d} \right)^2 \right]^{1/2} = \left[0.01^2 + \left(\frac{0.05 \cdot 20 \cdot 0.01}{19.974984} \right)^2 \right]^{1/2} = 0.010012$$

$$\phi_0 = \tan^{-1}\left(-\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d x_0} \right) = \tan^{-1}\left(-\frac{0.05 \cdot 20}{19.974984} \right) = -2.865984^\circ$$

Thank You

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