BEE2143 – Signals & Networks

Chapter 7 – Laplace Transform

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Definition of Laplace Transform

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Definition of Laplace Transform

- Laplace transform is another method to transform a signal from time domain to frequency domain (s-domain)
- The basic idea of Laplace transform comes from the Fourier transform
- As we have seen in the previous chapter, not many functions have their Fourier transform such as t, t^2, e^t, etc.
- The Laplace transform formula is the modification of the Fourier transform formula:

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt,$$

  that is, the term $j\omega$ is replaced by s
- s is equal to $\sigma + j\omega$, where σ is a large positive real number

- The Laplace transform formula:

$$F(s) = \mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

- However, the Laplace transform only support the function $f(t)$ which domain $t \geq 0$
- In order for $f(t)$ to have a Laplace transform, the integral must converge to a finite value
- Since $|e^{j\omega t}| = 1$ for any value of t, the integral converges when

$$\int_{0^-}^{\infty} e^{-\sigma t}|f(t)|dt < \infty$$

Example 15.1, pg. 678 (Alexander & Sadiku, 2009)

Determine the Laplace transform of each of the following functions:

(a) $u(t)$

(b) $e^{-at}u(t), \; a > 0$

(c) $\delta(t)$

Answer:

(a) $\dfrac{1}{s}$

(b) $\dfrac{1}{s+a}$

(c) 1

- Comparison between Laplace transform and Fourier transform:

Laplace transform	**Fourier transform**
– One-sided (the integral is over $0 < t < \infty$), making it only useful for positive time functions, $f(t),\ t > 0$	– Applicable to functions defined for all time
– Applicable to a wider range of functions	– Exist for signals that are not physically realizable and have no Laplace transforms
– Better suited for the analysis of transient problems involving initial conditions, since it permits the inclusion of the initial conditions	– Specially useful for problems in the steady state

Laplace Transform Pairs

f(t)	***F(s)***
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$
$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t + \theta)u(t)$	$\frac{s \sin \theta + \omega \cos \theta}{s^2+\omega^2}$
$\cos(\omega t + \theta)u(t)$	$\frac{s \cos \theta - \omega \sin \theta}{s^2+\omega^2}$
$e^{-at} \sin \omega t u(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Properties of Laplace Transform

- Linearity

$$\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$$

- Time scaling

$$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

- Time shifting

$$\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$$

- Frequency shifting

$$\mathcal{L}[e^{-at}f(t)] = F(s+a)$$

- Time differentiation

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\left[\frac{d^nf(t)}{dt^n}\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$

- Time integration

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$$

- Frequency differentiation

$$\mathcal{L}[t^nf(t)] = (-1)^n\frac{d^nF(s)}{ds^n}$$

- Frequency integration

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(s)ds$$

- Time periodicity

$$\mathcal{L}[f(t)] = \frac{F_1(s)}{1 - e^{-sT}}, \quad f(t) = f(t+T)$$

- Initial value

$$f(0) = \lim_{s\to\infty} sF(s)$$

- Final value

$$f(\infty) = \lim_{s\to 0} sF(s)$$

- Convolution

$$\mathcal{L}[f(t) * g(t)] = F(s)G(s)$$

Practice Problem 15.3, pg. 687 (Alexander & Sadiku, 2009)

Find the Laplace transform of $f(t) = (\cos 3t + e^{-5t})u(t)$.

Answer:

$$F(s) = \frac{2s^2 + 5s + 9}{(s+5)(s^2+9)}$$

Practice Problem 15.4, pg. 688 (Alexander & Sadiku, 2009)

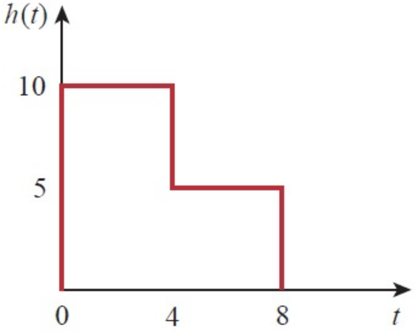
Determine the Laplace transform of $f(t) = t^2 \cos 3t u(t)$.

Answer:

$$F(s) = \frac{2s(s^2 - 27)}{(s^2 + 9)^3}$$

Practice Problem 15.5, pg. 688 (Alexander & Sadiku, 2009)

Find the Laplace transform of the function $h(t)$ in Fig. 15.6.

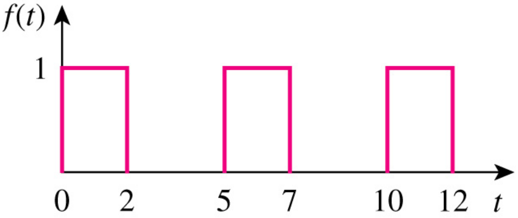


Answer:

$$H(s) = \frac{5}{s}(2 - e^{-4s} - e^{-8s})$$

Practice Problem 15.6, pg. 689 (Alexander & Sadiku, 2009)

Determine the Laplace transform of the periodic function in Fig. 15.8.



Answer:

$$F(s) = \frac{1 - e^{-2s}}{s(1 - e^{-5s})})$$

Practice Problem 15.7, pg. 690 (Alexander & Sadiku, 2009)

Obtain the initial and the final values of

$$G(s) = \frac{3s^3 + 2s + 10}{s(s+2)^2(s+3)}$$

Answer: 3, 0.83333

Inverse Laplace Transform

- The inverse Laplace transform is defined as

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{j2\pi}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}ds$$

  where the integration is performed along a straight line $(\sigma_1 + j\omega,\ -\infty < \omega < \infty)$ in the region of convergence, $\sigma_1 > \sigma_c$
- The direct application of this equation involves some knowledge about complex analysis
- For this reason, we will not use this equation to find the inverse Laplace transform
- We will rather use a look-up table of the Laplace transform pairs

- Suppose $F(s)$ has the general form of

$$F(s) = \frac{N(s)}{D(s)}$$

  where $N(s)$ is the numerator polynomial and $D(s)$ is the denominator polynomial
- The roots of $N(s)$ are called the zeros of $F(s)$, while the roots of $D(s)$ are the poles of $F(s)$
- We use partial fraction expansion to break $F(s)$ down into simple terms whose inverse transform we obtain from the Laplace transform pairs table
- Thus, finding the inverse Laplace transform of $F(s)$ involves two steps

- Steps to Find the Inverse Laplace Transform:
  1. Decompose $F(s)$ into simple terms using partial fraction expansion.
  2. Find the inverse of each term by matching entries in Laplace transform pairs table
- The three possible forms $F(s)$ may take:
  1. Simple poles
  $$F(s) = \frac{A}{s+a} + \frac{B}{s+b} + \dots$$
  2. Repeated poles
  $$F(s) = \frac{A_n}{(s+p)^n} + \dots + \frac{A_2}{(s+p)^2} + \frac{A_1}{s+p}$$
  3. Complex poles
  $$F(s) = \frac{A_1 s + A_2}{s^2 + bs + c} + \dots$$

Practice Problem 15.10, pg. 695 (Alexander & Sadiku, 2009)

Obtain $g(t)$ if

$$G(s) = \frac{s^3 + 2s + 6}{s(s+1)^2(s+3)}$$

Answer:

$$g(t) = (2 - 3.25e^{-t} - 1.5te^{-t} + 2.25e^{-3t})u(t)$$

Practice Problem 15.11, pg. 697 (Alexander & Sadiku, 2009)

Find $g(t)$ given that

$$G(s) = \frac{10}{(s+1)(s^2+4s+13)}$$

Answer:

$$g(t) = \left(e^{-t} - e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t\right)u(t)$$

Problem 15.37, pg. 712 (Alexander & Sadiku, 2009)

Find the inverse Laplace transform of:

(a) $H(s) = \dfrac{s+4}{s(s+2)}$

(b) $G(s) = \dfrac{s^2+4s+5}{(s+3)(s^2+2s+2)}$

(c) $F(s) = \dfrac{e^{-4s}}{s+2}$

(d) $D(s) = \dfrac{10s}{(s^2+1)(s^2+4)}$

Problem 15.37, pg. 712 (cont.)

Answer:

(a) $h(t) = (2 - e^{-2t})u(t)$

(b) $g(t) = (0.4e^{3t} + 0.6e^{-t}\cos t + 0.8e^{-t}\sin t)u(t)$

(c) $f(t) = e^{-2(t-4)}u(t-4)$

(d) $d(t) = \dfrac{10}{3}(\cos t - \cos 2t)u(t)$

Problem 15.48, pg. 713 (Alexander & Sadiku, 2009)

Find $f(t)$ using convolution given that:

(a) $F(s) = \dfrac{4}{(s^2 + 2s + 5)^2}$

(b) $F(s) = \dfrac{2s}{(s + 1)(s^2 + 4)}$

Answer:

(a) $f(t) = e^{-t}(0.25 \sin 2t - 0.5t \cos 2t)u(t)$

(b) $f(t) = (-0.4e^{-t} + 0.4 \cos 2t + 0.8 \sin 2t)u(t)$

List of References

1. C.K. Alexander and M.N.O. Sadiku (2009), Fundamentals of Electric Circuits 4th ed., New York: McGraw-Hill.