BEE2143 – Signals & Networks

Chapter 5 – Fourier Transform

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Definition and Properties of Fourier Transform

Fourier Transform using derivative technique

Inverse Fourier Transform

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Definition and Properties of Fourier Transform

- Fourier transform is another method to transform a signal from time domain to frequency domain
- The basic idea of Fourier transform comes from the complex Fourier series
- Practically, many signals are non-periodic
- Fourier transform use the principal of the Fourier series, with assumption that the period of the non-periodic signal is infinity $(T \to \infty)$

- Definition of the Fourier transform:

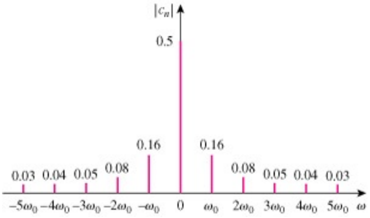
$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

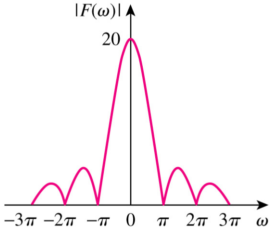
- Generally, the Fourier transform $F(\omega)$ exists when the Fourier integral converges
- A *sufficient but not necessary* condition for a function $f(t)$ to have a Fourier transform is, it can be completely integrable, i.e.

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty$$

- Comparison between Fourier series and Fourier transform:

Fourier series	**Fourier transform**
– Support periodic function	– Support non-periodic function
– Discrete frequency spectrum	– Continuous frequency spectrum

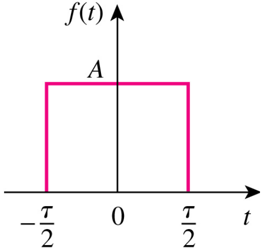




f(t)	**F(ω)**
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$u(t+\tau) - u(t-\tau)$	$2\frac{\sin\omega\tau}{\omega}$
$\|t\|$	$-\frac{2}{\omega^2}$
$\text{sgn}(t)$	$\frac{2}{j\omega}$
$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
$e^{at}u(-t)$	$\frac{1}{a-j\omega}$
$t^n e^{-at}u(t)$	$\frac{n!}{(a+j\omega)^{n+1}}$
$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$
$\cos\omega_0 t$	$\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$
$e^{-at}\sin\omega_0 t u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$
$e^{-at}\cos\omega_0 t u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$

Example 18.2, pg. 815 (Alexander & Sadiku, 2009)

Derive the Fourier transform of a single rectangular pulse of width $\tau = 2$ and height $A = 10$, shown in Fig. 18.4.

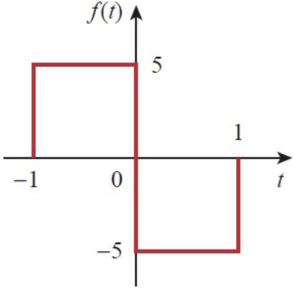


Answer:

$$F(\omega) = 20\,\mathrm{sinc}\,\omega$$

Practice Problem 18.2, pg. 815 (Alexander & Sadiku, 2009)

Obtain the Fourier transform of the function in Fig. 18.6.

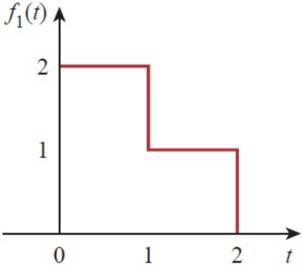


Answer:

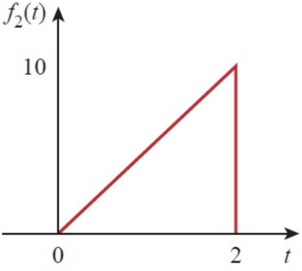
$$F(\omega) = \frac{10(\cos\omega - 1)}{j\omega}$$

Problem 18.7, pg. 842 (Alexander & Sadiku, 2009)

Find the Fourier transforms of the signals in Fig. 18.32.



(a)



(b)

Answer:

$$F_1(\omega) = \frac{2 - e^{-j\omega} - e^{-j2\omega}}{j\omega}, \quad F_2(\omega) = \frac{5e^{-j2\omega}}{\omega^2}(1 + j2\omega) - \frac{5}{\omega^2}$$

Properties of the Fourier transform:

- Linearity

$$\mathcal{F}[af(t) + bg(t)] = aF(\omega) + bG(\omega)$$

- Time scaling

$$\mathcal{F}[f(at)] = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

- Time shifting

$$\mathcal{F}[f(t-a)] = e^{-j\omega a}F(\omega)$$

- Frequency shifting

$$\mathcal{F}[e^{jat}f(t)] = F(\omega - a)$$

- Amplitude modulation

$$\mathcal{F}[f(t)\cos\omega_0 t] = \frac{1}{2}F(\omega-\omega_0) + \frac{1}{2}F(\omega+\omega_0)$$

Properties of the Fourier transform (cont.):

- Time differentiation

$$\mathcal{F}\left[\frac{d^n f(t)}{dt^n}\right] = (j\omega)^n F(\omega)$$

- Time integration

$$\mathcal{F}\left[\int_{-\infty}^{t} f(t)dt\right] = \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

- Frequency differentiation

$$\mathcal{F}[t^n f(t)] = j^n \frac{d^n F(\omega)}{d\omega^n}$$

- Reversal

$$\mathcal{F}[f(-t)] = F(-\omega) = F^*(\omega)$$

Properties of the Fourier transform (cont.):

- Duality

$$\mathcal{F}[F(t)] = 2\pi f(-\omega)$$

- Convolution in t

$$\mathcal{F}[f(t) * g(t)] = F(\omega)G(\omega)$$

- Convolution in ω

$$\mathcal{F}[f(t)g(t)] = \frac{1}{2\pi}F(\omega) * G(\omega)$$

Problem 18.23, pg. 843 (Alexander & Sadiku, 2009)

If the Fourier transform of $f(t)$ is

$$\frac{10}{(2+j\omega)(5+j\omega)}$$

determine the transforms of the following:

(a) $f(-3t)$ (b) $f(2t-1)$ (c) $f(t)\cos 2t$

(d) $\dfrac{df(t)}{dt}$ (e) $\displaystyle\int_{-\infty}^{t} f(t)dt$

Problem 18.23, pg. 843 (cont.)

Answer:

(a) $$\frac{30}{(6-j\omega)(15-j\omega)}$$

(b) $$\frac{20e^{-j\omega/2}}{(4+j\omega)(10+j\omega)}$$

(c) $$\frac{5}{[2+j(\omega+2)][5+j(\omega+2)]}+\frac{5}{[2+j(\omega-2)][5+j(\omega-2)]}$$

(d) $$\frac{j10\omega}{(2+j\omega)(5+j\omega)}$$

(e) $$\frac{10}{j\omega(2+j\omega)(5+j\omega)}+\pi\delta(\omega)$$

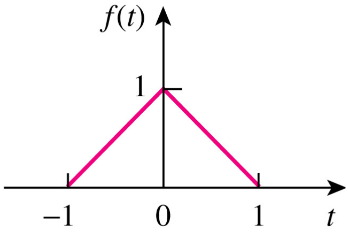
Fourier Transform using derivative technique

- The simplest Fourier transform is on the delta function, where $\mathcal{F}[\delta(t)] = 1$
- Using this idea, before we transformed a function, we differentiate it until its derivative is expressed in delta functions form
- The important properties in implementing this technique are:

$$\mathcal{F}\left[\frac{d^n f(t)}{dt^n}\right] = (j\omega)^n F(\omega) \quad \text{and} \quad \mathcal{F}[\delta(t-a)] = e^{-j\omega a}$$

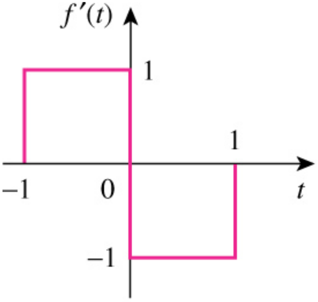
Example 18.5, pg. 827 (Alexander & Sadiku, 2009)

Find the Fourier transform of the function in Fig. 18.14.

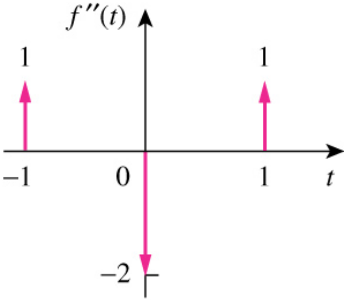


Example 18.5, pg. 827 (cont.)

Answer:



(a)

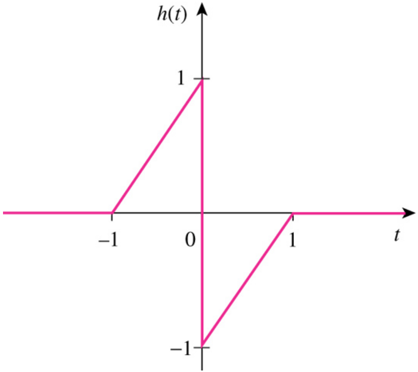


(b)

$$F(\omega) = \frac{2(1 - \cos\omega)}{\omega^2}$$

Problem 18.5, pg. 841 (Alexander & Sadiku, 2009)

Obtain the Fourier transform of the signal shown in Fig. 18.30.



Answer:

$$H(\omega) = \frac{j2}{\omega} - \frac{j2}{\omega^2}\sin\omega$$

Inverse Fourier Transform

- The definition of Fourier transform is

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

- The inverse Fourier transform is defined as

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

- Note that the function $f(t)$ and its transform $F(\omega)$ can be derived from each other

The principle of duality

- If we interchange t and ω such as

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{j\omega t}dt$$

and replace ω with $-\omega$ such as

$$f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{-j\omega t}dt,$$

we have

$$\mathcal{F}[F(t)] = 2\pi f(-\omega)$$

- This is an important property to find the Fourier transform of certain functions which their Fourier integral diverges

Example 18.6, pg. 828 (Alexander & Sadiku, 2009)

Obtain the inverse Fourier transform of:

(a) $F(\omega) = \dfrac{10j\omega + 4}{(j\omega)^2 + 6j\omega + 8}$

(b) $G(\omega) = \dfrac{\omega^2 + 21}{\omega^2 + 9}$

Answer:

(a) $f(t) = (18e^{-4t} - 8e^{-2t})u(t)$

(b) $g(t) = \delta(t) + 2e^{-3|t|}$

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Problem 18.27, pg. 844 (Alexander & Sadiku, 2009)

Find the inverse Fourier transforms of the following functions:

(a) $F(\omega) = \dfrac{100}{j\omega(j\omega + 10)}$

(b) $G(\omega) = \dfrac{10j\omega}{(-j\omega + 2)(j\omega + 3)}$

(c) $H(\omega) = \dfrac{60}{-\omega^2 + j40\omega + 1300}$

(d) $Y(\omega) = \dfrac{\delta(\omega)}{(j\omega + 1)(j\omega + 2)}$

Problem 18.27, pg. 844 (cont.)

Answer:

(a) $f(t) = 5\,\mathrm{sgn}(t) - 10e^{-10t}u(t)$

(b) $g(t) = 4e^{2t}u(-t) - 6e^{-3t}u(t)$

(c) $h(t) = 2e^{-20t}\sin 30tu(t)$

(d) $y(t) = \dfrac{1}{4\pi}$

Problem 18.28, pg. 844 (Alexander & Sadiku, 2009)

Find the inverse Fourier transforms of:

(a) $$\frac{\pi\delta(\omega)}{(5+j\omega)(2+j\omega)}$$

(b) $$\frac{10\delta(\omega+2)}{j\omega(j\omega+1)}$$

(c) $$\frac{20\delta(\omega-1)}{(2+j\omega)(3+j\omega)}$$

(d) $$\frac{5\pi\delta(\omega)}{5+j\omega}+\frac{5}{j\omega(5+j\omega)}$$

Problem 18.28, pg. 844 (cont.)

Answer:

(a) $\dfrac{1}{20}$

(b) $-\dfrac{5e^{-j2t}}{2\pi(4+j)}$

(c) $\dfrac{2e^{jt}}{\pi(1+j)}$

(d) $(1-e^{-5t})u(t)$

List of References

1. C.K. Alexander and M.N.O. Sadiku (2009), Fundamentals of Electric Circuits 4th ed., New York: McGraw-Hill.