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CHAPTER 3

HYPOTHESIS TESTING

Expected Outcomes

- ✓ Able to test a population mean when population variance is known or unknown.
- ✓ Able to test the difference between two populations mean when population variances are known or unknown.
- ✓ Able to test paired data using *z*-test and *t*-test.
- ✓ Able to test population proportion using *z*-test.
- ✓ Able to test the difference between two populations proportion using *z*-test.
- ✓ Able to test a population variance and test the difference between two populations variances.
- ✓ Able to determine the relationship between hypothesis testing and confidence interval.
- ✓ Able to solve hypothesis testing using Microsoft Excel.

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3.2 Test Hypothesis for Population Mean with known and unknown Population Variance
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3.5 Test Hypotheses for Population Proportion
3.6 Test Hypotheses for the Difference between Two Population Proportions
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3.10 Relationship between Hypothesis Tests and Confidence Interval

3.1 INTRODUCTION TO HYPOTHESIS TESTING

- A **statistical hypothesis** is a statement or conjecture or assertion concerning a parameter or parameters of one or more populations. Many problems in science and engineering require that we need to decide either to accept or reject a statement about some parameter, which is a decision-making process for evaluating claims or statement about the population(s). The decision-making procedure about the hypothesis is called **hypothesis testing**.

3 Methods of Hypothesis Testing

- **The traditional method**
- The *P* - value method
- The confidence interval method

3.1.1 TERMS AND DEFINITION

Definition 1a:

A **null hypothesis**, denoted by is a statistical hypothesis that states an assertion about one or more population parameters.

Definition 1b:

The **alternative hypothesis** denoted by is a statistical hypothesis that states the assertion of all situations that not covered by the null hypothesis.

$H_0 : \theta = \theta_0$	**TWO TAILED TEST**	$H_1 : \theta \neq \theta_0$
$H_0 : \theta \leq \theta_0$	**RIGHT TAILED TEST**	$H_1 : \theta > \theta_0$
$H_0 : \theta \geq \theta_0$	**LEFT TAILED TEST**	$H_1 : \theta < \theta_0$

parameter (θ) A value (θ_0)

Types Of Hypothesis

Type of Hypothesis		Hypothesis
Two-tailed test		$H_0 : \theta = \theta_0$ $H_1 : \theta \neq \theta_0$
One-tailed test	Right-tailed test	$H_0 : \theta \leq \theta_0$ $H_1 : \theta > \theta_0$
	Left-tailed test	$H_0 : \theta \geq \theta_0$ $H_1 : \theta < \theta_0$

Note:

(i) The H_0 should have 'equals' sign and H_1 should not have 'equals' sign.
(ii) The H_0 is on trial and always initially assumed to be true.
(iii) Accept H_0 if the sample data are consistent with the null hypothesis.
(iv) Reject H_0 if the sample data are inconsistent with the null hypothesis, and accept the alternative hypothesis.

Definition 2: A **test statistic** is a sample statistic computed from the data obtained by random sampling.

→ $Z_{\text{test}}, t_{\text{test}}, \chi^2_{\text{test}}, f_{\text{test}}$

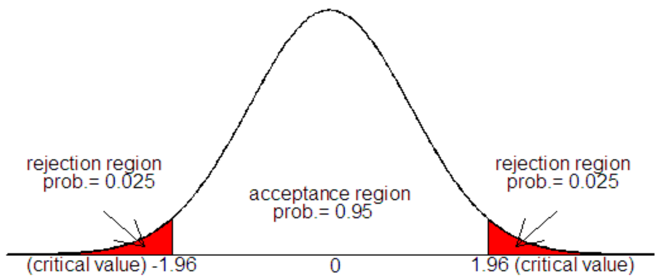
Definition 3: The **rejection (critical) region** α, is the set of values for the test statistics that leads to rejection of the null hypothesis.

Definition 4: The **acceptance region**, $1 - \alpha$ is the set of values for the test statistics that leads to acceptance of the null hypothesis.

Definition 5: The **critical value(s)** is the value(s) of boundary that separate the rejection and acceptance regions.

Definition 6: The **decision rule** of a statistical hypothesis test is a rule that specifies the conditions under which the null hypothesis may be rejected.

→ Reject H_0 if test statistics **>** critical value

<table>
<tr><th colspan="2">Type of Test</th><th>Hypothesis</th><th>Rejection Region</th><th>Graphical Display
(Hypothesis using test statistic z with $\alpha = 0.05$)</th></tr>
<tr><td colspan="2">Two-tailed test</td><td>$H_0 : \theta = \theta_0$
$H_1 : \theta \neq \theta_0$</td><td>Both sides</td><td>
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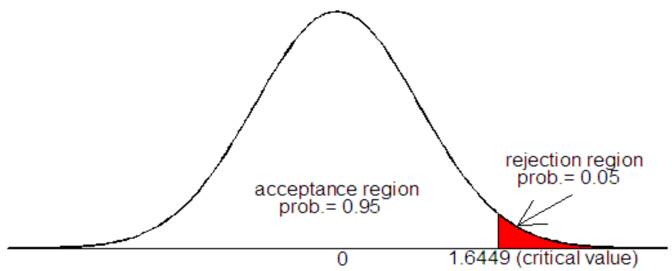
<tr><td rowspan="2">One-tailed test</td><td>Right-tailed test</td><td>$H_0 : \theta \leq \theta_0$
$H_1 : \theta > \theta_0$</td><td>Right side</td><td>
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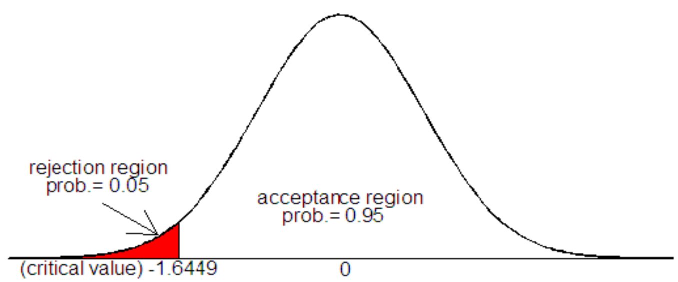
<tr><td>Left-tailed test</td><td>$H_0 : \theta \geq \theta_0$
$H_1 : \theta < \theta_0$</td><td>Left side</td><td>
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Definition 7: Rejecting the null hypothesis when it is true is defined as Type I error.

→ P(Type I error) = α (significance level)

Definition 8: Failing to reject the null hypothesis when it is false in state of nature is defined as Type II error.

→ P(Type II error) = β

Possible Outcomes:

State of Nature	Statistical Conclusion/decision	
	Reject H_0	Not to reject H_0
H_0 is true	Type I error	Correct decision
H_0 is false	Correct decision	Type II error

Example 2

The additive might not significantly increase the lifetimes of automobile batteries in the population, but it might increase the lifetime of the batteries in the sample. In this case, H_0 would be rejected when it was really true, which committing a **type I error**.

While, the additive might not work on the batteries selected for the sample, but if it were to be used in the general population of batteries, it might significantly increase their lifetime. Hence based on the information obtained from the sample, would not reject the H_0, thus committing a **type II error**.

Hypothesis testing common phrase

$> \;:\; H_1$	$< \;:\; H_1$
Is greater than	Is less than
Is above	Is below
Is higher than	Is lower than
Is longer than	Is shorter than
Is bigger than	Is smaller than
Is increased	Is decreased or reduced from
$\geq \;:\; H_0$	$\leq \;:\; H_0$
Is greater than or equal	Is less than or equal
Is at least	Is at most
Is not less than	Is not more than
$= \;:\; H_0$	$\neq \;:\; H_1$
Is equal to	Is not equal to
Is exactly the same as	Is different from
Has not changed from	Has changed from
Is the same as	Is not the same as

3.2.1 PROCEDURES OF HYPOTHESIS TESTING

Step 1: Formulate a **hypothesis** and **state the claim**

Two-tailed test	OR	Right-tailed test	OR	Left-tailed test
$H_0 : \theta = \theta_0$		$H_0 : \theta \le \theta_0$		$H_0 : \theta \ge \theta_0$
$H_1 : \theta \ne \theta_0$		$H_1 : \theta > \theta_0$		$H_1 : \theta < \theta_0$

Step 2: Choose the appropriate **test statistic**, and calculate the sample test statistic value: $Z_{\text{test}}, t_{\text{test}}, \chi^2_{\text{test}}, f_{\text{test}}$

Step 3: Establish the test criterion by determining the **critical value** (point) and **critical region**

- Significance level value, α
- Inequality $(\ne, >, <)$ used in the H_1

Step 4: Make a **decision** to reject or not to reject the H_0.

Step 5: Draw a **conclusion** to reject or to accept the claim or statement.

Hypothesis Testing: Step by Step

Extract all given information

↓

Define Parameter

↓

Define H_0 and H_1

↓

Choose appropriate test statistics

↓

Find critical value

↓

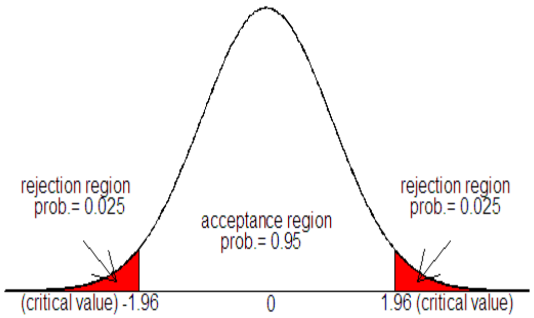
Test the hypothesis (rejection region)

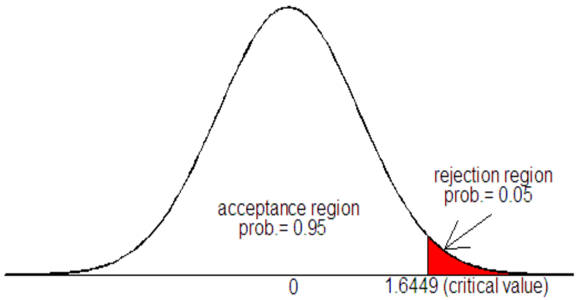
↓

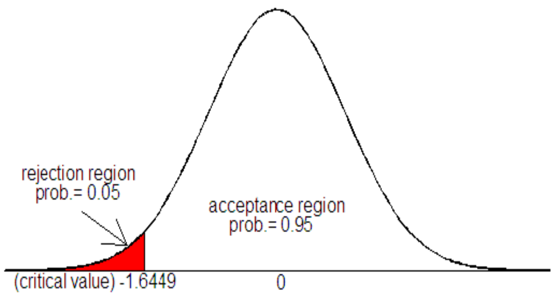
Make a conclusion – there is enough evidence to reject/accept the claim at α

3.2: TEST HYPOTHESES FOR POPULATION MEAN, μ WITH KNOWN AND UNKNOWN POPULATION VARIANCE

Two-tailed test	Right-tailed test	Left-tailed test
$H_0 : \mu = \mu_0$	$H_0 : \mu \le \mu_0$	$H_0 : \mu \ge \mu_0$
$H_1 : \mu \ne \mu_0$	$H_1 : \mu > \mu_0$	$H_1 : \mu < \mu_0$







Test Statistics of Hypothesis Testing for Mean μ

- σ^2 is known

$$z_{test} = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$$

- σ^2 is unknown
  - $n \geq 30$

$$z_{test} = \frac{\bar{X} - \mu_0}{s / \sqrt{n}}$$

  - $n < 30$

$$t_{test} = \frac{\bar{X} - \mu_0}{s / \sqrt{n}}$$

Where: $\mu_0 = population\ mean$

NOTE: $\boldsymbol{Z_{test}}$ and $\boldsymbol{t_{test}}$ are test statistics

The Rejection Criteria (1)

i. If the population variance, σ^2 is **known**, the test statistic to be used is

$$z_{\text{test}} = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \sim z_\alpha .$$

Therefore the rejection procedure for each type of hypothesis can be summarised as in following **Table**

Hypothesis testing for μ with known σ^2

H_0	H_1	**Statistical Test**	**Reject H_0 if**
$H_0 : \mu = \mu_0$	$H_1 : \mu \neq \mu_0$	$z_{\text{test}} = \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$	$z_{test} > z_{\alpha/2}$ or $z_{test} < -z_{\alpha/2}$
$H_0 : \mu \leq \mu_0$	$H_1 : \mu > \mu_0$		$z_{test} > z_\alpha$
$H_0 : \mu \geq \mu_0$	$H_1 : \mu < \mu_0$		$z_{test} < -z_\alpha$

The Rejection Criteria (2)

ii. If the population variance, σ^2 is **unknown** and the **sample size is large**, i.e. $n \geq 30$, then the test statistic to be used is

$$z_{\text{test}} = \frac{\bar{x} - \mu_0}{s / \sqrt{n}} \sim z_\alpha \;.$$

Therefore the rejection procedure for each type of hypothesis can be summarised as in following **Table**

Hypothesis testing for μ with unknown σ^2 and $n \geq 30$

H_0	H_1	**Statistical Test**	**Reject H_0 if**
$H_0 : \mu = \mu_0$	$H_1 : \mu \neq \mu_0$		$z_{test} > z_{\alpha/2}$ or $z_{test} < -z_{\alpha/2}$
$H_0 : \mu \leq \mu_0$	$H_1 : \mu > \mu_0$	$z_{\text{test}} = \dfrac{\bar{x} - \mu_0}{s / \sqrt{n}}$	$z_{test} > z_\alpha$
$H_0 : \mu \geq \mu_0$	$H_1 : \mu < \mu_0$		$z_{test} < -z_\alpha$

The Rejection Criteria (3)

iii. If the population variance, σ^2 is **unknown** and the **sample size is small**, i.e. $n < 30$, then the test statistic to be used is

$$t_{\text{test}} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \sim t_{\alpha,v} \quad \text{where } v = n - 1$$

Therefore the rejection procedure for each type of hypothesis can be summarised as in following **Table**

Hypothesis testing for μ with unknown σ^2 and $n < 30$

H_0	H_1	**Statistical Test**	**Reject H_0 if**
$H_0 : \mu = \mu_0$	$H_1 : \mu \neq \mu_0$	$t_{\text{test}} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$	$t_{test} > t_{\alpha/2,n-1}$ or $t_{test} < -t_{\alpha/2,n-1}$
$H_0 : \mu \leq \mu_0$	$H_1 : \mu > \mu_0$		$t_{test} > t_{\alpha,n-1}$
$H_0 : \mu \geq \mu_0$	$H_1 : \mu < \mu_0$		$t_{test} < -t_{\alpha,n-1}$

Example 3

Most water-treatment facilities monitor the quality of their drinking water on an hourly basis. One variable monitored is pH, which measures the degree of alkalinity or acidity in the water. A pH below 7.0 is acidic, above 7.0 is alkaline and 7.0 is neutral. One water-treatment plant has target a pH of 8.5 (most try to maintain a slightly alkaline level). The mean and standard deviation of 1 hour's test results based on 31 water samples at this plant are 8.42 and 0.16 respectively. **Does this sample provide sufficient evidence that the mean pH level in the water differs from 8.5?** Use a 0.05 level of significance. Assume that the population is approximately normally distributed.

Solution:

Step 1: *Formulate a* ***hypothesis*** *and* ***state the claim***.

X: pH level in the water

$$H_0 : \mu = 8.5$$

$$H_1 : \mu \neq 8.5 \quad (claim)$$

Example 3: solution

Step 2: *Choose the appropriate* ***test statistic*** *and calculate the sample test statistic value.*

Since σ^2 is unknown, i.e. $s^2 = 0.16^2$ and $n \geq 30$,

the test statistic is $z_{\text{test}} = \dfrac{\bar{x} - \mu_0}{s / \sqrt{n}} = \dfrac{8.42 - 8.5}{0.16 / \sqrt{31}} = -2.7839$.

Step 3: *Establish the test criterion by determining the* ***critical value*** *and* ***rejection region****.*

H_0	H_1	**Statistical Test**	**Reject H_0 if**
$H_0 : \mu = \mu_0$	$H_1 : \mu \neq \mu_0$	$z_{\text{test}} = \dfrac{\bar{x} - \mu_0}{s / \sqrt{n}}$	$z_{test} > z_{\alpha/2}$ or $z_{test} < -z_{\alpha/2}$
$H_0 : \mu \leq \mu_0$	$H_1 : \mu > \mu_0$		$z_{test} > z_{\alpha}$
$H_0 : \mu \geq \mu_0$	$H_1 : \mu < \mu_0$		$z_{test} < -z_{\alpha}$

Example 3: solution

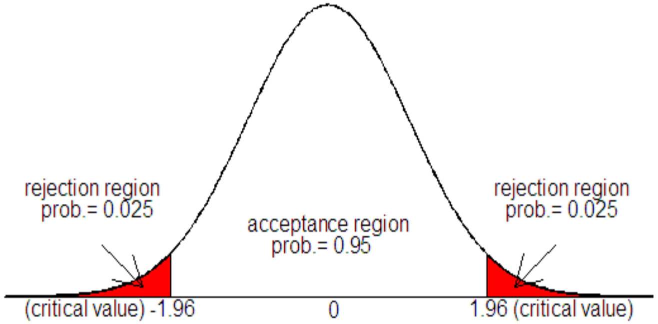
Step 3: *Establish the test criterion by determining the* ***critical value*** *and* ***rejection region***.

H_0	H_1	**Statistical Test**	**Reject H_0 if**
$H_0 : \mu = \mu_0$	$H_1 : \mu \neq \mu_o$	$z_{\text{test}} = \dfrac{\overline{x} - \mu_0}{s / \sqrt{n}}$	$z_{test} > z_{\alpha/2}$ or $z_{test} < -z_{\alpha/2}$

Given $\alpha = 0.05$ and the test is two-tailed test, hence the critical values are $z_{0.025} = 1.9600$ and $-z_{0.025} = -1.9600$.

Step 4: *Make a* ***decision*** *to reject or fail to reject the* H_0.

Since $\left(z_{\text{test}} = -2.7839\right) < \left(-1.96 = -z_{0.025}\right)$, then we reject H_0.



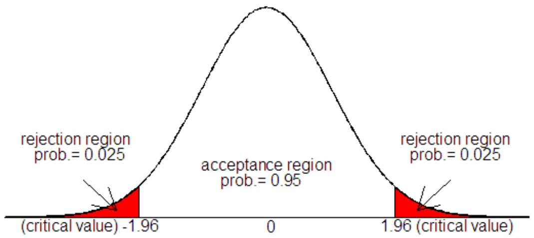
Step 5: *Draw a* ***conclusion*** *to reject or to accept the claim or statement.*

At $\alpha = 0.05$, the sample provide sufficient evidence that the mean pH level in the water differs from 8.5.

3.3 TEST HYPOTHESES FOR THE DIFFERENCE BETWEEN TWO POPULATIONS MEAN

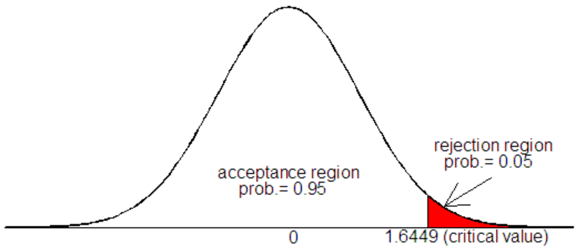
$$H_0 : \mu_1 - \mu_2 = \mu_0$$
$$H_1 : \mu_1 - \mu_2 \neq \mu_0$$

Two-tailed test



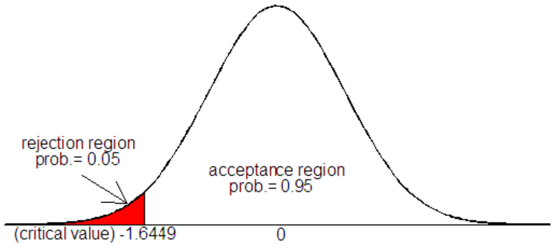
$$H_0 : \mu_1 - \mu_2 \leq \mu_0$$
$$H_1 : \mu_1 - \mu_2 > \mu_0$$

Right-tailed test



$$H_0 : \mu_1 - \mu_2 \geq \mu_0$$
$$H_1 : \mu_1 - \mu_2 < \mu_0$$

Left-tailed test



Test Statistics for the Difference between Means

Hypothesis Testing for $\mu_1 - \mu_2$

- σ_1^2 and σ_2^2 are known,
  $$z_{test} = \frac{(\bar{x}_1 - \bar{x}_2) - \mu_o}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim z_\alpha$$
- σ_1^2 and σ_2^2 are unknown,
  - $\sigma_1^2 = \sigma_2^2$
    - $n_1 \geq 30, n_2 \geq 30$
      $$z_{test} = \frac{(\bar{x}_1 - \bar{x}_2) - \mu_o}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim z_\alpha$$
    - $n_1 < 30, n_2 < 30$
      $$t_{test} = \frac{(\bar{x}_1 - \bar{x}_2) - \mu_o}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{\alpha, n_1+n_2-2}$$
  - $\sigma_1^2 \neq \sigma_2^2$
    - $n_1 \geq 30, n_2 \geq 30$
      $$z_{test} = \frac{(\bar{x}_1 - \bar{x}_2) - \mu_o}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \sim z_\alpha$$
    - $n_1 < 30, n_2 < 30$
      $$t_{test} = \frac{(\bar{x}_1 - \bar{x}_2) - \mu_o}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \sim t_{\alpha, \upsilon}$$

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

$$\upsilon = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{\left(\frac{s_1^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{s_2^2}{n_2}\right)^2}{n_2 - 1}}$$

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Example 4

The overall distance travelled of a golf ball is tested by hitting the ball with the golf stick. Ten balls selected randomly from two different brands are tested and the overall distance is measured and the data is given as follows.

Overall distance travelled of golf ball (in meters)

Brand 1	251	262	263	248	259	248	255	251	240	244
Brand 2	236	223	238	242	250	257	248	247	240	245

By assuming that both population variances are unequal, can we say that both brands of ball have similar average overall distance? Use $\alpha = 0.05$.

Example 4: solution

Step 1:

X_1: Overall distance travelled of golf ball from brand 1
X_2: Overall distance travelled of golf ball from brand 2

The hypothesis is

$$H_0 : \mu_1 - \mu_2 = 0 \quad (claim)$$
$$H_1 : \mu_1 - \mu_2 \neq 0$$

Step 2:

Statistic	Brand 1	Brand 2
n	10	10
$\bar{x}$	252.1	242.6
s	7.6077	9.2640

Since σ_1^2 and σ_2^2 are **unknown,** $\sigma_1^2 \neq \sigma_2^2$, and $n_1 < 30, n_2 < 30$, then the test statistic is

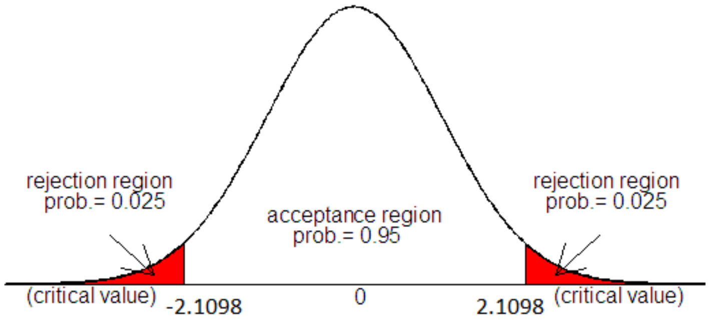
$$t_{test} = \frac{(\bar{x}_1 - \bar{x}_2) - \mu_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{(252.1 - 242.6) - 0}{\sqrt{\frac{7.6077^2}{10} + \frac{9.2640^2}{10}}} = 2.5061$$

Example 4: solution

Step 3: Given $\alpha = 0.05$ and the test is two-tailed test. The critical value is

$$t_{\frac{\alpha}{2},\, \nu} = t_{0.025,\, 17} = 2.1098$$

where $$\nu = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{\left(\frac{s_1^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{s_2^2}{n_2}\right)^2}{n_2 - 1}} = \frac{\left(\frac{7.6077^2}{10} + \frac{9.2640^2}{10}\right)^2}{\frac{\left(\frac{7.6077^2}{10}\right)^2}{9} + \frac{\left(\frac{9.2640^2}{10}\right)^2}{9}} = 17.3442 \approx 17$$



Step 4: Since $(t_{test} = 2.5061) > (t_{0.025,\, 17} = 2.1098)$, H_0 is rejected.

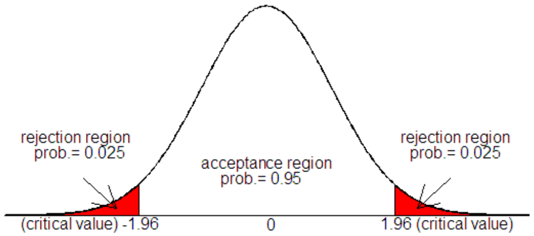
Step 5: At $\alpha = 0.05$, there is no significant evidence to support that both brands of ball have similar average overall distance

3.4 TEST HYPOTHESES FOR PAIRED DATA

Two-tailed test

$$H_0 : \mu_D = \mu_0$$

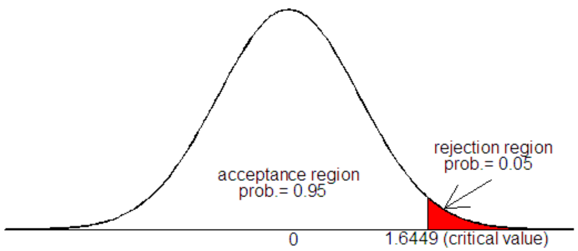
$$H_1 : \mu_D \neq \mu_0$$



Right-tailed test

$$H_0 : \mu_D \leq \mu_0$$

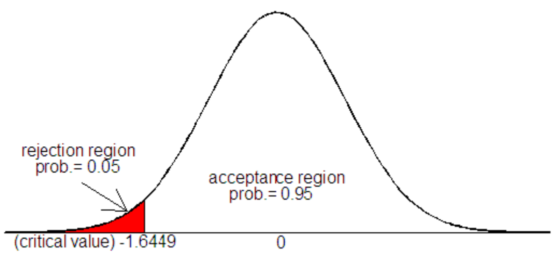
$$H_1 : \mu_D > \mu_0$$



Left-tailed test

$$H_0 : \mu_D \geq \mu_0$$

$$H_1 : \mu_D < \mu_0$$



Test Statistics

$$t_{test} = \frac{\bar{x}_D - \mu_0}{s_D / \sqrt{n}} \sim t_{\alpha,v}$$

where

$D = X_1 - X_2$ differences between the paired sample

n is number of paired sample

$\bar{x}_D, s_D$, are the mean and standard deviation for the difference of paired sample, respectively

$v = n - 1$, degrees of freedom

$\mu_D = \mu_1 - \mu_2$ is population mean difference, $-\infty < \mu_D < \infty$

Example 5

A new gadget is installed to air conditioner unit(s) in a factory to minimize the number of bacteria floating in the air. The number of bacteria floating in the air before and after the installation for a week in the factory is recorded as follows.

Before	**10.1**	**11.6**	**12.1**	**9.1**	**10.3**	**15.3**	**13.0**
After	11.2	8.5	8.4	8.4	8.0	7.6	7.2

Is it wise for the factory management to install the new gadget? By assuming the data is approximately normally distributed, test the hypothesis at 5% level of significance.

Example 5: solution

$H_0 : \mu_D \le 0$

$H_1 : \mu_D > 0$ (wise to install the new gadget)

where $\mu_D = \mu_{\text{Before-After}}$

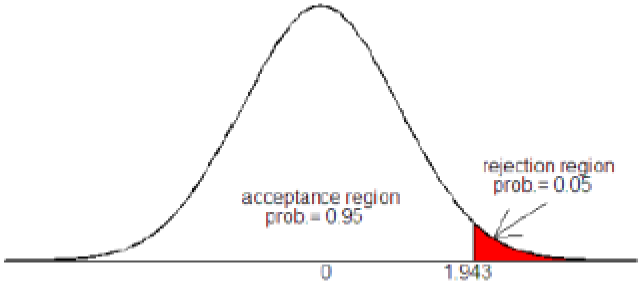
Before, X_1	10.1	11.6	12.1	9.1	10.3	15.3	13.0
After, X_2	11.2	8.5	8.4	8.4	8.0	7.6	7.2
$D = X_1 - X_2$	-1.1	3.1	3.7	0.7	2.3	7.7	5.8

$\bar{x}_D = 3.1714$

$s_D = 2.9669$

$t_{0.05,6} = 1.943$

$$t_{test} = \frac{3.1714 - 0}{2.9669/\sqrt{7}} = 2.8281$$



Since $(t_{test} = 2.8281) > (t_{0.05,6} = 1.943)$, H_0 is rejected.

It is wise for the factory management to install the new gadget, at 5% level of significance.

3.5 TEST HYPOTHESES FOR POPULATION PROPORTION

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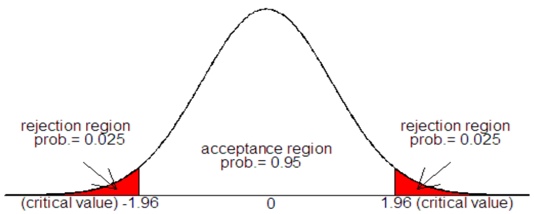
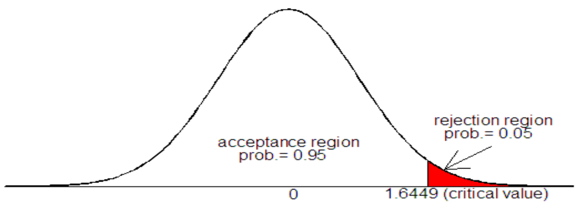
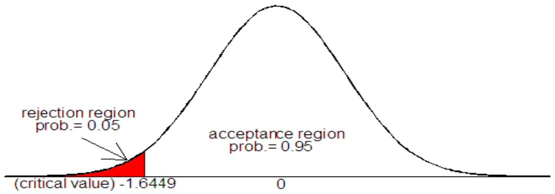
■ The hypothesis:

Two-tailed test		Right-tailed test		Left-tailed test
$H_0 : \pi = \pi_0$ $H_1 : \pi \neq \pi_0$	OR	$H_0 : \pi \leq \pi_0$ $H_1 : \pi > \pi_0$	OR	$H_0 : \pi \geq \pi_0$ $H_1 : \pi < \pi_0$







■ The Test Statistics:

$$z_{test} = \frac{p - \pi_0}{\sqrt{\frac{\pi_0 (1 - \pi_0)}{n}}} \sim z_\alpha$$

where

- $p = \frac{x}{n}$ - sample proportion
- π_0 - given population proportion

Example 6

An attorney claims that at least 25% of all lawyers advertise. A sample of 200 lawyers in a certain city showed that 63 had used some form of advertising. At $\alpha = 0.05$, is there enough evidence to support the attorney's claim?

Solution:

Step 1: X is the number of lawyers advertise

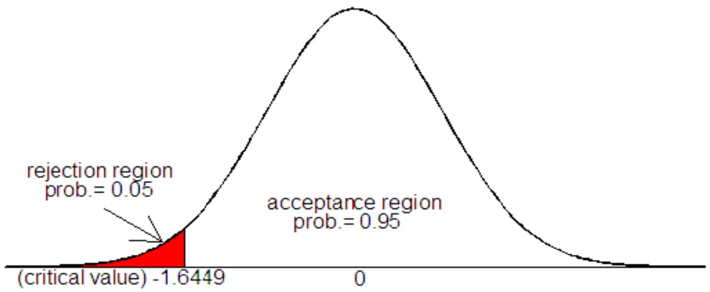
$$H_0 : \pi \geq 0.25 \quad (claim)$$
$$H_1 : \pi < 0.25$$

Example 6: solution

Step 2: Since $n = 200$ and $x = 63$, then $p = \dfrac{63}{200} = 0.315$.

The test statistic is $z_{\text{test}} = \dfrac{0.315 - 0.25}{\sqrt{\dfrac{(0.25)(0.75)}{200}}} = 2.1229$.

Step 3: Given $\alpha = 0.05$ and the test is left-tailed test, hence the critical value is $-z_{0.05} = -1.6449$.



Step 4: Since $(z_{\text{test}} = 2.1229) > (-1.6449 = -z_{0.05})$, then we accept H_0.

Step 5: At $\alpha = 0.05$, there is enough evidence to support the attorney's claim.

3.6 TEST HYPOTHESES FOR DIFFERENCE BETWEEN TWO POPULATIONS PROPORTION

- **The hypothesis:**

Type of Test	Hypothesis	Decision on Rejection
Two-tailed test	$H_0: \pi_1 - \pi_2 = \pi_0$ $H_1: \pi_1 - \pi_2 \neq \pi_0$	Reject H_0 if $z_{\text{test}} < -z_{\alpha/2}$ or $z_{\text{test}} > z_{\alpha/2}$
Right-tailed test	$H_0: \pi_1 - \pi_2 \leq \pi_0$ $H_1: \pi_1 - \pi_2 > \pi_0$	Reject H_0 if $z_{\text{test}} > z_\alpha$
Left-tailed test	$H_0: \pi_1 - \pi_2 \geq \pi_0$ $H_1: \pi_1 - \pi_2 < \pi_0$	Reject H_0 if $z_{\text{test}} < -z_\alpha$

- **The Test Statistics:**

If $\pi_0 \neq 0$:

$$z_{test} = \frac{(p_1 - p_2) - \pi_0}{\sqrt{\frac{\pi_1(1-\pi_1)}{n_1} + \frac{\pi_2(1-\pi_2)}{n_2}}} \sim z_\alpha$$

If $\pi_0 = 0$:

$$z_{\text{test}} = \frac{(p_1 - p_2)}{\sqrt{p_p(1-p_p)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim z_\alpha \quad \text{where } p_p = \frac{X_1 + X_2}{n_1 + n_2}$$

Example 7

An experiment was conducted in order to determine whether the increased levels of carbon dioxide (CO_2) will kill the leaf-eating insects. Two containers, labeled X and Y were filled with two levels of CO_2. Container Y had double of CO_2 level compared to container X. Assume that 80 insect larvae were placed at random in each container. After two days, the percentage of larvae that died in container X and Y were five percent and ten percent, respectively. Do these experimental results demonstrate that an increased level of CO_2 is effective in killing leaf-eating insects' larvae? Test at 1% significance level.

Example 7: solution

Step 1:

X: the number of the number of larvae that died in container X
Y: the number of the number of larvae that died in container Y

$$H_0 : \pi_Y - \pi_X \le 0$$
$$H_1 : \pi_Y - \pi_X > 0 \ (claim)$$

Step 2:

Statistic	Y	X
n	80	80
p	0.1	0.05
x	8	4

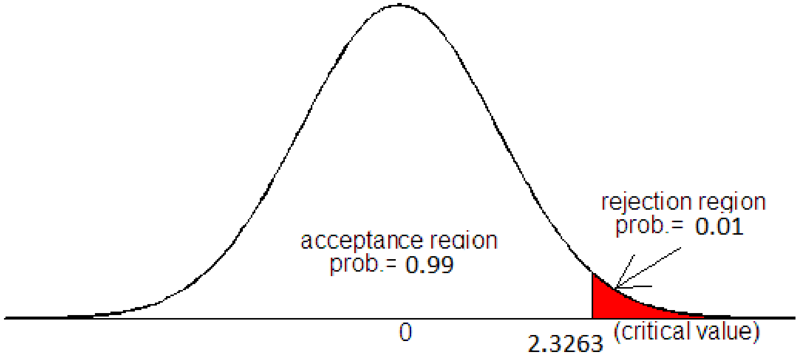
The test statistic is

$$z_{test} = \frac{(p_Y - p_X) - \pi_0}{\sqrt{P_p\left(1 - P_p\right)\left(\frac{1}{n_Y} + \frac{1}{n_X}\right)}} = \frac{\left(0.1 - 0.05\right) - 0}{\sqrt{0.075\left(1 - 0.075\right)\left(\frac{1}{80} + \frac{1}{80}\right)}} = 1.2006$$

where $P_p = \frac{x_Y + x_X}{n_Y + n_X} = \frac{8+4}{80+80} = 0.075$

Example 7: solution

Step 3: Given $\alpha = 0.01$ and the test is right-tailed test, hence the critical value is $z_{0.01} = 2.3263$.



Step 4: Since $\left(z_{test} = 1.2006\right) < \left(z_{0.01} = 2.3263\right)$, then we failed to reject H_0.

Step 5: At $\alpha = 0.01$, there is no significant evidence to support that an increased level of carbon dioxide is effective in killing higher percentage of leaf-eating insects' larvae.

3.7 TEST HYPOTHESES FOR A POPULATION VARIANCE

- **The hypothesis:**

Type of Test Hypothesis	Decision on Rejection Reject H_0 if	
Two-tailed test $H_0 : \sigma^2 = \sigma_0^2$ $H_1 : \sigma^2 \neq \sigma_0^2$	$\chi^2_{\text{test}} < \chi^2_{1-\alpha/2, n-1}$ or $\chi^2_{\text{test}} > \chi^2_{\alpha/2, n-1}$	
Right-tailed test $H_0 : \sigma^2 \leq \sigma_0^2$ $H_1 : \sigma^2 > \sigma_0^2$	$\chi^2_{\text{test}} > \chi^2_{\alpha, n-1}$	
Left-tailed test $H_0 : \sigma^2 \geq \sigma_0^2$ $H_1 : \sigma^2 < \sigma_0^2$	$\chi^2_{\text{test}} < \chi^2_{1-\alpha, n-1}$	

- **The Test Statistics:**

$$\chi^2_{test} = \frac{(n-1)s^2}{\sigma_0^2} \sim \chi^2_{\alpha, v=n-1}$$

s^2 is the sample variance, σ_0^2 is the given variance

Example 8

Listed below are waiting times (in minutes) of customers at a bank.

6.5	6.8	7.1	7.3	7.4	7.7

The management will open more teller windows if the standard deviation of waiting times (in minutes) is at least 0.9 minutes. Is there enough evidence to open more teller windows at $\alpha = 0.01$?

Example 8: solution

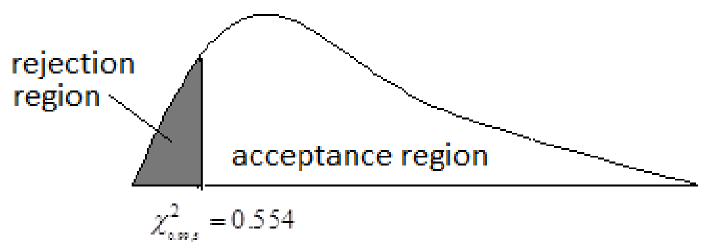
Step 1: X is waiting times (in minutes) of customers at a bank

$$H_0 : \sigma^2 \geq 0.9^2 \text{ minutes} \qquad \text{(open more teller windows)}$$
$$H_1 : \sigma^2 < 0.9^2 \text{ minutes}$$

Step 2: $n = 6$ customers $\quad \bar{x} = 7.13$ minutes $\quad s = 0.43$ minutes

The test statistic is $\chi^2_{test} = \dfrac{(n-1)s^2}{\sigma_0} = \dfrac{(6-1)0.43^2}{0.9^2} = 1.1414$

Step 3: Given $\alpha = 0.01$ and the test is left-tailed test, hence the critical value is $\chi^2_{0.99,5} = 0.554$.



Step 4: Since $\left(\chi^2_{test} = 1.1414\right) > \left(\chi^2_{0.99,5} = 0.554\right)$, then we failed to reject H_0.

Step 5: At $\alpha = 0.01$, there is enough evidence to open more teller windows.

3.8 TEST HYPOTHESES FOR THE RATIO OF TWO POPULATION VARIANCES

- **The hypothesis:**

Type of Test	Hypothesis	Decision on Rejection
Two-tailed test	$H_0: \sigma_1^2 = \sigma_2^2$ $H_1: \sigma_1^2 \neq \sigma_2^2$	Reject H_0 if $f_{\text{test}} < f_{1-\alpha/2, n_1-1, n_2-1}$ or $f_{\text{test}} > f_{\alpha/2, n_1-1, n_2-1}$ where $f_{1-\frac{\alpha}{2}, n_1-1, n_2-1} = \dfrac{1}{f_{\frac{\alpha}{2}, n_2-1, n_1-1}}$
Right-tailed test	$H_0: \sigma_1^2 \leq \sigma_2^2$ $H_1: \sigma_1^2 > \sigma_2^2$	Reject H_0 if $f_{\text{test}} > f_{\alpha, n_1-1, n_2-1}$
Left-tailed test	$H_0: \sigma_1^2 \geq \sigma_2^2$ $H_1: \sigma_1^2 < \sigma_2^2$	Reject H_0 if $f_{\text{test}} < f_{1-\alpha, n_1-1, n_2-1}$ where $f_{1-\alpha, n_1-1, n_2-1} = \dfrac{1}{f_{\alpha, n_2-1, n_1-1}}$

- **The Test Statistics:**

$$f_{test} = \frac{s_1^2}{s_2^2} \sim f_{v_1, v_2} \quad \text{where} \quad v_1 = n_1 - 1, \quad v_2 = n_2 - 1$$

Example 9

A manager of computer operations of a large company wants to study the computer usage of two departments within the company. The departments are Human Resource Department and Research Department. The processing time (in seconds) for each job is recorded as follows:

Human Resource	9	3	8	7	12	
Research	4	13	10	9	9	6

Is there any difference in the variability of processing times for the two departments at $\alpha = 0.05$.

Example 9: solution

Step 1: X_1: processing time (in seconds) for each jobs from for Human Resource Department
X_2: processing time (in seconds) for each jobs from for Research Department

$$H_0 : \sigma_1^2 = \sigma_2^2 \qquad (claim)$$
$$H_1 : \sigma_1^2 \neq \sigma_2^2$$

Step 2: $n_1 = 5, \quad \bar{x}_1 = 7.8, \quad s_1 = 3.3$
$n_2 = 6, \quad \bar{x}_1 = 8.5, \quad s_1 = 3.1$

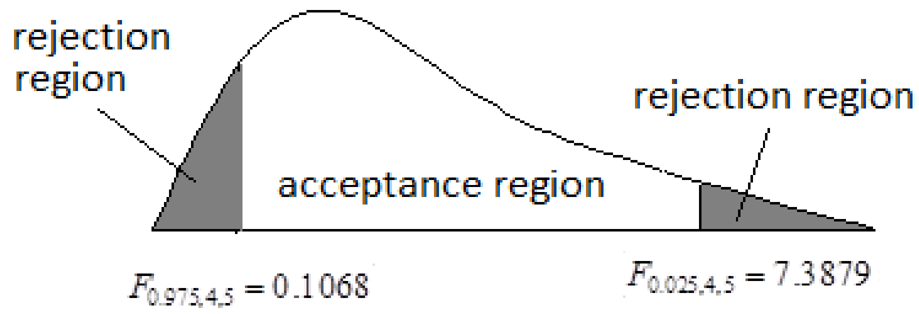
The test statistic is $F_{test} = \dfrac{s_1^2}{s_2^2} = \dfrac{3.3^2}{3.1^2} = 1.1332$

Step 3: Given $\alpha = 0.05$ and the test is two-tailed test, hence the critical value are

$$F_{\frac{\alpha}{2}, n_1-1, n_2-1} = F_{0.025,4,5} = 7.3879$$

$$F_{1-\frac{\alpha}{2}, n_1-1, n_2-1} = \frac{1}{F_{\frac{\alpha}{2}, n_2-1, n_1-1}} = F_{0.975,4,5} = \frac{1}{F_{0.025,5,4}} = \frac{1}{9.3645} = 0.1068$$

Example 9: solution



Step 4: Since $\left(F_{0.975,4,5} = 0.1068\right) < \left(f_{test} = 1.1332\right) < \left(F_{0.025,4,5} = 7.3879\right)$, then we failed to reject H_0.

Step 5: At $\alpha = 0.05$, there is no difference in the variability of processing times for the two departments.

3.9 *P*-Values IN HYPOTHESIS TESTING

The ***P*-value** (Probability value) is the smallest level of significance that would lead to rejection of the null hypothesis with the given data

- **Finding the P-value**

Statistical Table	Calculator (*Casio fx-570 MS*)
Step 1: Find the area under the standard normal distribution curve corresponding to the *z* test value. **Step 2:** Subtracting the area from 0.5 to get the *P*-value for a right-tailed or left-tailed test. To get the *P*-value for a two-tailed test, double the area after subtracting.	**Step 1:** Find the area under the standard normal distribution curve corresponding to the *z* test value. **Step 2:** The area obtained is the *P*-value for a right-tailed or left-tailed test. To get the *P*-value for a two-tailed test, double the area. $P\text{-value} = P(Z > 1.6449) = \mathrm{R}(1.6449) = 0.05$

Procedures of Hypothesis Testing using P-Value Approach

Step 1: Formulate a **hypothesis** and **state the claim**

Two-tailed test	Right-tailed test	Left-tailed test
$H_0 : \theta = \theta_0$	$H_0 : \theta \leq \theta_0$	$H_0 : \theta \geq \theta_0$
$H_1 : \theta \neq \theta_0$	$H_1 : \theta > \theta_0$	$H_1 : \theta < \theta_0$

Step 2: Choose the appropriate **test statistic**, and calculate the sample test statistic value.

Step 3: Find the P-value

Step 4: Make a **decision** to reject or not to reject the H_0

$$\text{If } P-\text{value} \leq \alpha \Rightarrow \text{ Reject } H_0$$

$$\text{If } P-\text{value} > \alpha \Rightarrow \text{ Do not Reject } H_0$$

Step 5: Draw a **conclusion** to reject or to accept the claim or statement.

Example 10

Most water-treatment facilities monitor the quality of their drinking water on an hourly basis. One variable monitored is pH, which measures the degree of alkalinity or acidity in the water. A pH below 7.0 is acidic, above 7.0 is alkaline and 7.0 is neutral. One water-treatment plant has target a pH of 8.5 (most try to maintain a slightly alkaline level). The mean and standard deviation of 1 hour's test results based on 31 water samples at this plant are 8.42 and 0.16 respectively. Does this sample provide sufficient evidence that the mean pH level in the water differs from 8.5? Use a 0.05 level of significance. Assume that the population is approximately normally distributed. **[Example 3]**

Solve this problem using P-value approach.

Example 11: solution

By considering Example 3.3, the P-value is calculated manually as follows.

Step 1: Formulate a **hypothesis** and **state the claim**.

$$H_0 : \mu = 8.5$$
$$H_1 : \mu \neq 8.5$$

Step 2: Choose the appropriate **test statistic** and calculate the sample test statistic value.

Since σ^2 is unknown, i.e. $s^2 = 0.16^2$, the test statistic is

$$z_{\text{test}} = \frac{8.42 - 8.5}{\frac{0.16}{\sqrt{31}}} = -2.7839$$

Step 3: Find the ***P*-value**. By using calculator

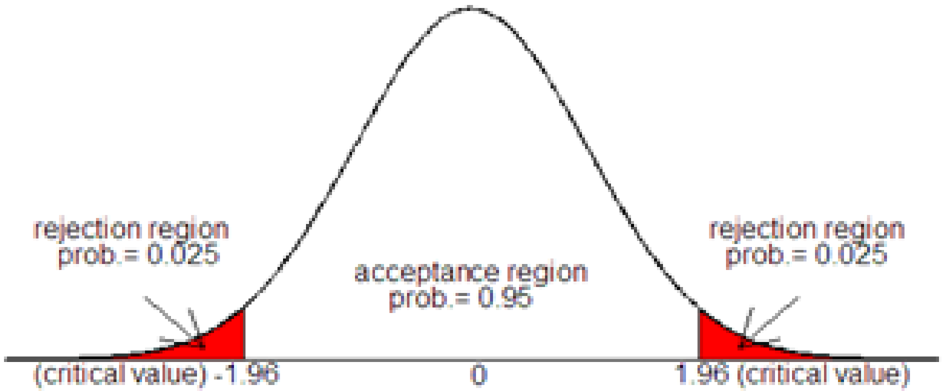
(1) area corresponding to the z test value (two-tailed test):
$$P(Z < -2.7839) + P(Z > 2.7839) = 2 \times R(2.7839) = 0.00538$$

(2) P-value $= 0.00538$

Example 11: solution

Step 4: Make a **decision** to reject or not to reject the H_0

Since $(P\text{-value} = 0.00538) < (\alpha = 0.05)$, then we reject H_0



Step 5: Draw a **conclusion** to reject or to accept the claim

At 10% significance level, the sample provide sufficient evidence that the mean pH level in the water differs from 8.5

P-value Using Excel – Test For Mean

Step 1:
Click Menu Data → Data Analysis →Descriptive Statistics → click OK

Step 2:
a) The commands for ***t*-test** are

(i) *t*-test = (Mean - μ_0)/Standard Error
(ii) *P*-value for a two-tailed test = T.DIST.2T(ABS(*t*-test), degrees of freedom)
P-value for a right-tailed test = T.DIST.RT((ABS(*t*-test), degrees of freedom)
P-value for a left-tailed test = T.DIST(ABS(*t*-test), degrees of freedom, 1)

***Note*:** Standard Error is a standard deviation divided by the square root of the number of data which can be written as $\text{s.e.} = \dfrac{\sigma}{\sqrt{n}}$.

Example 11

A petroleum company is studying to buy an additive for improving the distilled product. The company estimates the cost of the additive, which is RM1 million for 5 tonnes. Ten consultant companies submitted their tenders with the following estimates (in million RM):

0.97	**0.95**	**1.10**	**1.30**	**1.10**	**0.96**	**0.97**	**1.20**	**1.50**	**1.70**

Do you think the petroleum company over estimates the cost of the additive? Give your reason. Use P-value method.

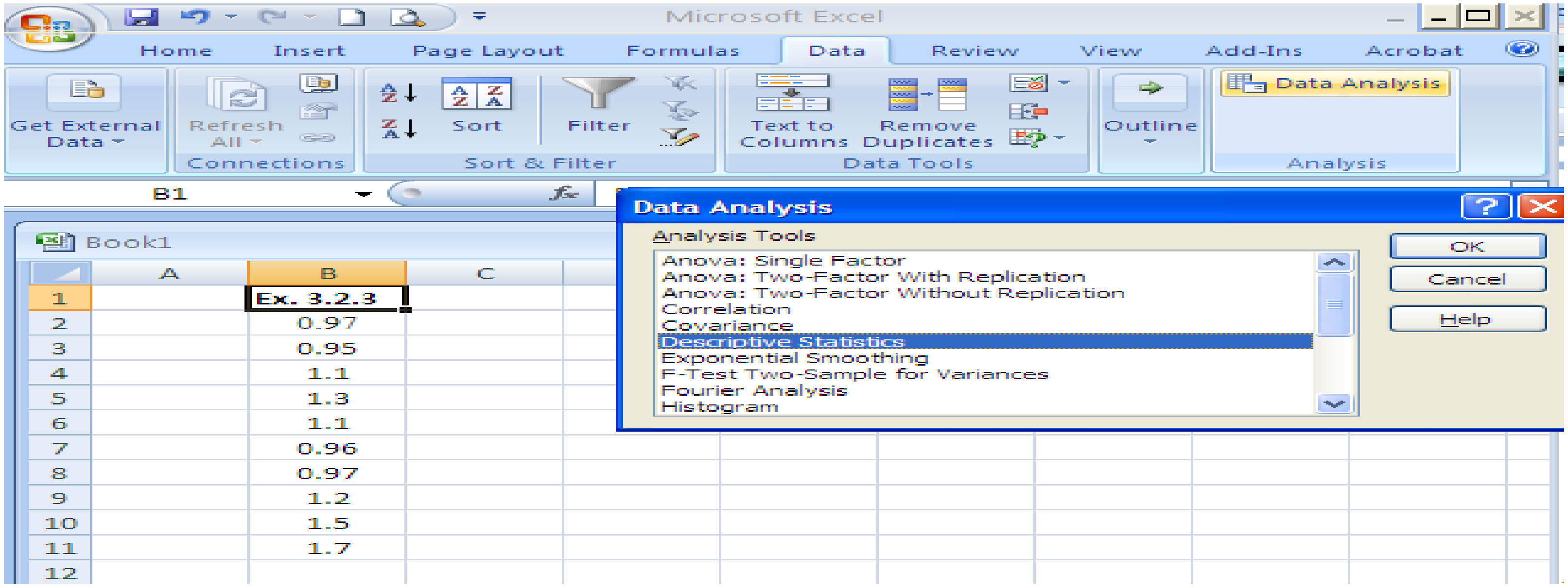
Example 11: solution

Step 1: ***Formulate*** *the hypothesis*

$$H_0 : \mu_C \geq 1$$

$$H_1 : \mu_C < 1 \text{ (claim: company over estimate the cost)}$$

Step 2: ***Key in*** *the data, select data → data analysis →Descriptive Statistics → click OK*



Example 11: solution

Output from *Excel*:

Column1	
Mean	**1.175**
Standard Error	**0.080942366**
Median	1.1
Mode	0.97
Standard Deviation	0.255962237
Sample Variance	0.065516667
Kurtosis	0.524938867
Skewness	1.172718741
Range	0.75
Minimum	0.95
Maximum	1.7
Sum	11.75
Count	10
Confidence Level(95.0%)	0.183104353
***t*-test**	**2.162032171**
***P*-value**	**0.970563811**

The values highlighted will be used to calculate *t*-test

t-test and *P*-value are calculated using *Excel* command as follows:

t-test = (1.175-1)/0.080942366

Since the case is *t*-test and left-tailed test,
P-value = T.DIST(2.162032171,9,1)

Step 3: $P\text{-value} = 0.9706$

Step 4: Since $(P\text{-value} = 0.9706) > (\alpha = 0.05)$, then we do not reject H_0.

Step 5: At $\alpha = 0.05$, there is not enough evidence to support the claim that the petroleum company over estimate the cost of the additive.

P-value Using Excel – Test For Difference Mean

Step 1: Test the difference in variability --> F.TEST(data set 1, data set 2)

Step 2: Click Menu Data--> Data Analysis--> Choose the appropriate test

(*i.e.: t-Test: Two-Sample Assuming Unequal Variances*)--> click ok

Step 3: Variable 1 range--> select the data set 1

Variable 2 range--> select the data set 2

Hypothesized mean difference--> value of μ_0

Alpha--> value of significance level, α

Step 4:

P-value for a two-tailed test = P(T<=t) two-tails (*depends on distribution used*)

P-value for a right-tailed test = P(T<=t) one-tail (*depends on distribution used*)

P-value for a left-tailed test = 1- P(T<=t) one-tail (*depends on distribution used*)

Example 12

A company is considering installing a new machine to assemble its product. The company is considering two types of machine, Machine A and Machine B but it will by only one machine. The company will install Machine B if the mean time taken to assemble a unit of the product is less than Machine A. Table below shows the time taken (in minutes) to assemble one unit of the product on each type of machine.

Machine A	23	26	19	24	27	22	20	18
Machine B	21	24	23	25	24	28	24	23

At 10% significance level, test the difference in variability between the two types of machines. Which machine should be installed by the company to assemble its product?

Example 12: solution

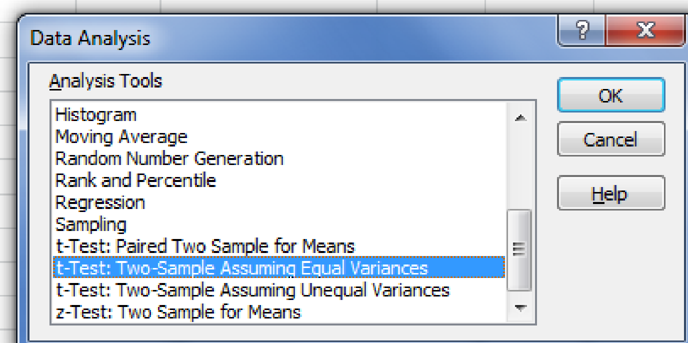
Step 1: ***Formulate** the hypothesis*

$$H_0 : \mu_A - \mu_B \le 0 \qquad H_0 : \sigma_A^2 = \sigma_B^2$$
$$H_1 : \mu_A - \mu_B > 0 \text{ (claim)} \quad \text{and} \quad H_1 : \sigma_A^2 \ne \sigma_B^2$$

machine A	Machine B
23	21
26	24
19	23
24	25
27	24
22	28
20	24
18	23

F-Test Two-Sample for Variances

	machine A	*Machine B*		
Mean	22.375	24		
Variance	10.55357	4		
Observations	8	8		
df	7	7		
F	2.638393			
P(F<=f) one-tail	0.111931			
F Critical one-tail	2.78493			
pvalue	0.223861	0.1	accept H0	



P-value =0.2239 > 0.1

Thus, Failed to reject H_0. There is no difference in the variability.

Example 12: solution

Step 2: *Key in the data in Excel and choose the t-Test: Two-Sample Assuming equal Variances*

t-Test: Two-Sample Assuming Equal Variances

	machine A	*Machine B*
Mean	22.375	24
Variance	10.55357	4
Observations	8	8
Pooled Variance	7.276786	
Hypothesized Mean Difference	0	
df	14	
t Stat	-1.2048	
P(T<=t) one-tail	0.124127	
t Critical one-tail	1.34503	
P(T<=t) two-tail	0.248254	
t Critical two-tail	1.76131	

Step 3: The test is one-tailed test, hence *P*-value = 0.1242

Step 4: Since $(P-\text{value}=0.1242)>(0.1=\alpha)$, then we do not reject H_0.

Step 5: At 10% significance level, machine A should be installed.

P-value Using Excel – Test For Paired Data

Step 1: Click Menu Data--> Data Analysis--> Choose the appropriate test

(*i.e.: t-Test: Paired Two Sample for Means*)--> click ok

Step 2: Variable 1 range--> select the data set 1

Variable 2 range--> select the data set 2

Hypothesized mean difference--> value of μ_0

Alpha--> value of significance level, α

Step 3:

P-value for a two-tailed test = P(T<=t) two-tails (*depends on distribution used*)

P-value for a right-tailed test = P(T<=t) one-tail (*depends on distribution used*)

P-value for a left-tailed test = 1- P(T<=t) one-tail (*depends on distribution used*)

Example 13: refer data example 5

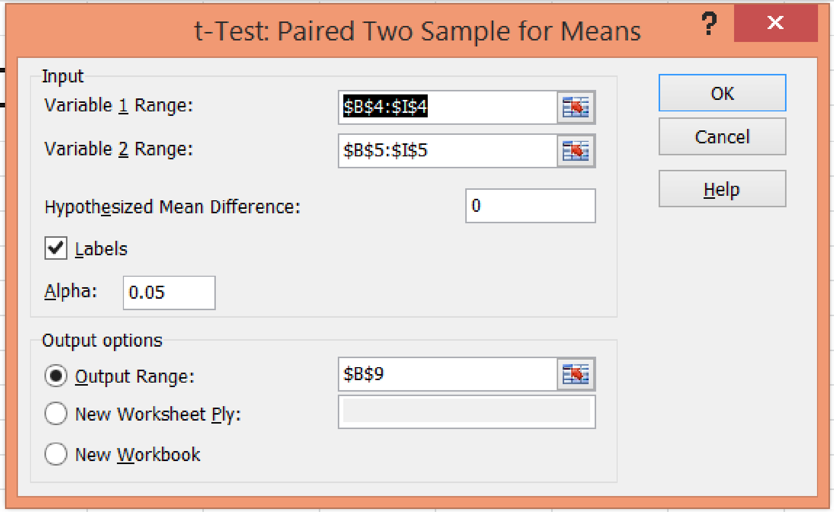
$H_0 : \mu_D \leq 0$

$H_1 : \mu_D > 0$ (wise to install the new gadget)

Before	**10.1**	**11.6**	**12.1**	**9.1**	**10.3**	**15.3**	**13**
After	11.2	8.5	8.4	8.4	8	7.6	7.2

t-Test: Paired Two Sample for Means

	Before	*After*
Mean	11.64286	8.471429
Variance	4.34619	1.675714
Observations	7	7
Pearson Correlation	-0.51515	
Hypothesized Mean Differe	0	
df	6	
t Stat	2.828159	
P(T<=t) one-tail	0.015015	
t Critical one-tail	1.94318	
P(T<=t) two-tail	0.03003	
t Critical two-tail	2.446912	



P-value =0.0150 < 0.05

Thus, reject *H*0. At 5% significance level, it is wise to install the new gadget.

3.10 RELATIONSHIP BETWEEN HYPOTHESIS TEST & CONFIDENCE INTERVAL

There is a relationship between the confidence interval and hypothesis test about the parameter, θ . Let say (a,b) is a $(1-\alpha)100\%$ confidence interval for the θ, the test of the size α of the hypothesis

$$H_0 : \theta = \theta_0$$
$$H_1 : \theta \neq \theta_0$$

will lead to rejection of H_0 if and only if θ_0 is not in the $(1-\alpha)100\%$ confidence interval (a,b).

Notes: This relationship should be checked for two-tailed test only.

Example 14

By considering *Example 3* again, the 95% confidence interval for μ is

$$= 8.42 \pm z_{0.025}\left(\frac{0.16}{\sqrt{31}}\right)$$

$$= 8.42 \pm 1.9600(0.0287)$$

$$= 8.42 \pm 0.0563$$

$$= (8.3637, 8.4763)$$

Since $\mu = 8.5$ is not included in this interval, the H_0 is rejected.
So, the decision making or conclusion is the same as in *Example 3*.

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Thank You

NEXT: CHAPTER 4 ANALYSIS OF VARIANCE