UMP OPEN COURSEWARE

Universiti Malaysia PAHANG

Engineering • Technology • Creativity

CHAPTER 4

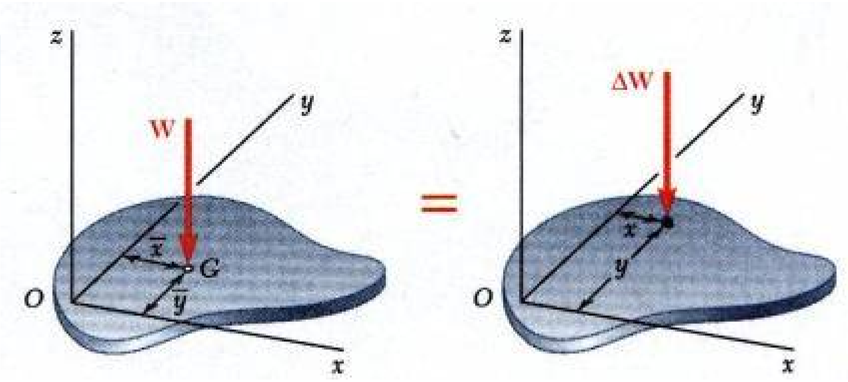
DISTRIBUTED FORCES: CENTROIDS AND CENTERS OF GRAVITY

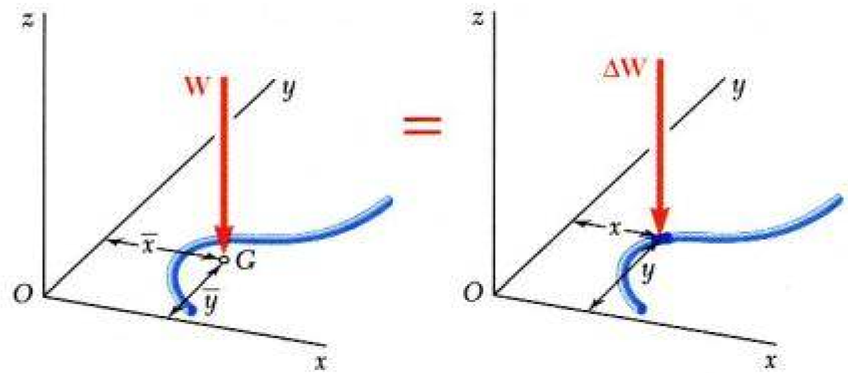
Expected Outcome:

- Able to determine centroids of composite areas and lines using the concept of the first moment of an area or a line
- Able to compute the area of a surface of revolution and of the volume of a body of revolution by using Theorem of Pappus Guldinus

Center of Gravity

- Center of gravity of a plate
- Center of gravity of a wire

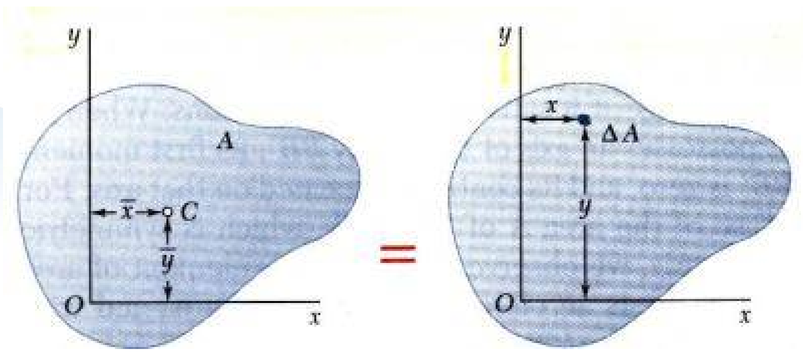




$$\sum M_y \quad \bar{x}W = \sum x\Delta W$$
$$= \int x\, dW$$
$$\sum M_x \quad \bar{y}W = \sum y\Delta W$$
$$= \int y\, dW$$

Centroids and First Moments of Areas and Lines

- Centroid of an area



$$\bar{x}W = \int x\,dW$$

$$\bar{x}(\gamma A t) = \int x(\gamma t)\,dA$$

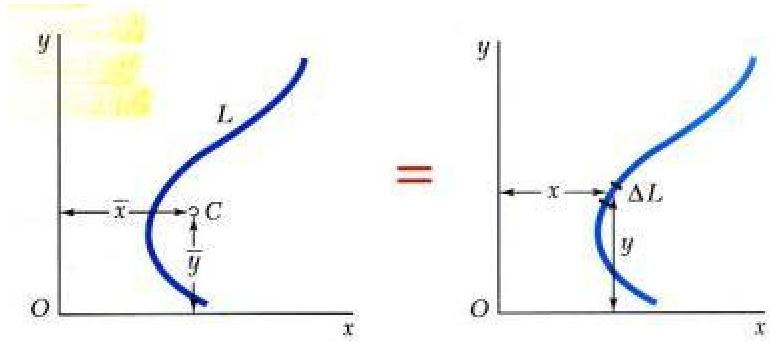
$$\bar{x}A = \int x\,dA = Q_y$$

$$= \text{first moment with respect to } y$$

$$\bar{y}A = \int y\,dA = Q_x$$

$$= \text{first moment with respect to } x$$

- Centroid of a line

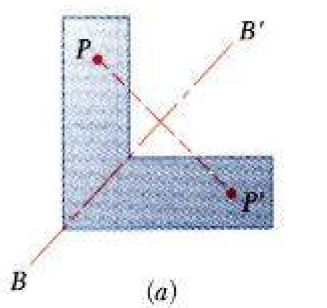


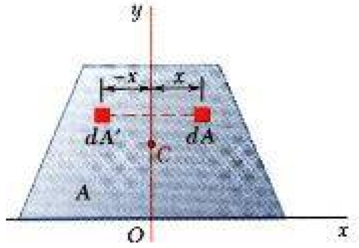
$$\bar{x}W = \int x\,dW$$

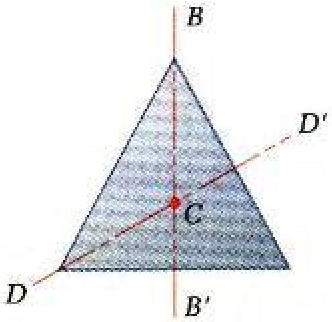
$$\bar{x}(\gamma L a) = \int x(\gamma a)\,dL$$

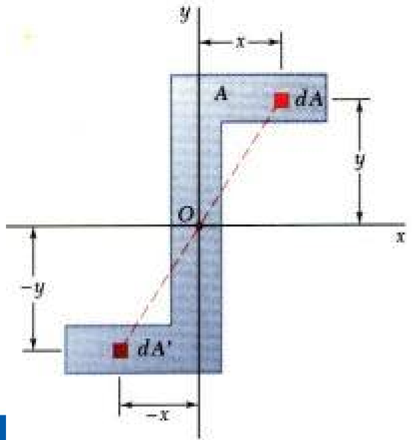
$$\bar{x}L = \int x\,dL$$

$$\bar{y}L = \int y\,dL$$



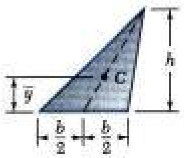
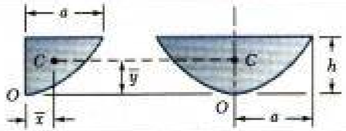
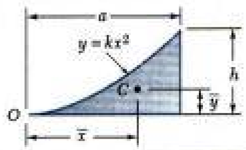
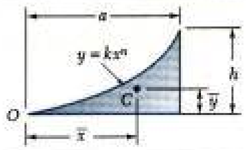
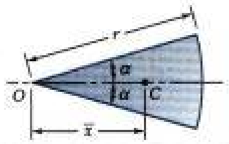




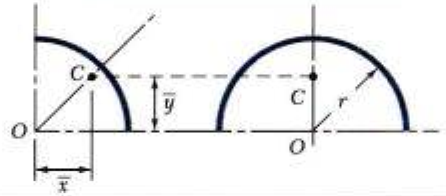
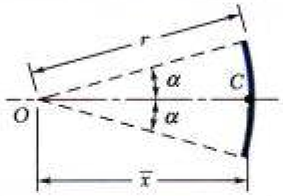


- An area is symmetric with respect to an axis *BB'* if for every point *P* there exists a point *P'* such that *PP'* is perpendicular to *BB'* and is divided into two equal parts by *BB'*.
- The first moment of an area with respect to a line of symmetry is zero.
- If an area possesses a line of symmetry, its centroid lies on that axis
- If an area possesses two lines of symmetry, its centroid lies at their intersection.
- An area is symmetric with respect to a center *O* if for every element *dA* at (*x*,*y*) there exists an area *dA'* of equal area at (-*x*,-*y*).
- The centroid of the area coincides with the center of symmetry.

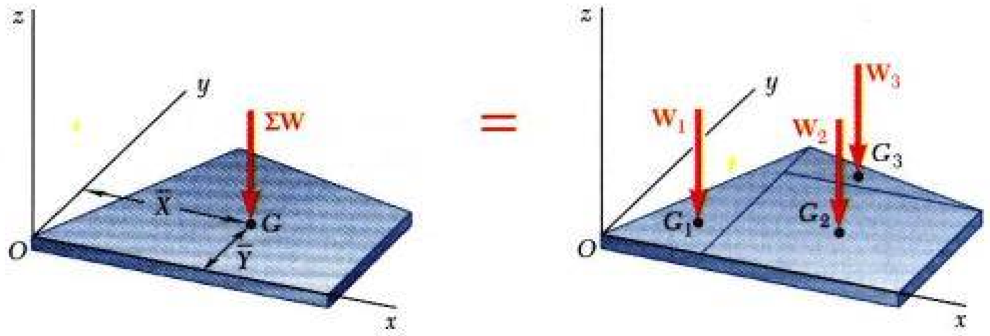
Centroids of Common Shapes of Areas

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2

Centroids of Common Shapes of Lines

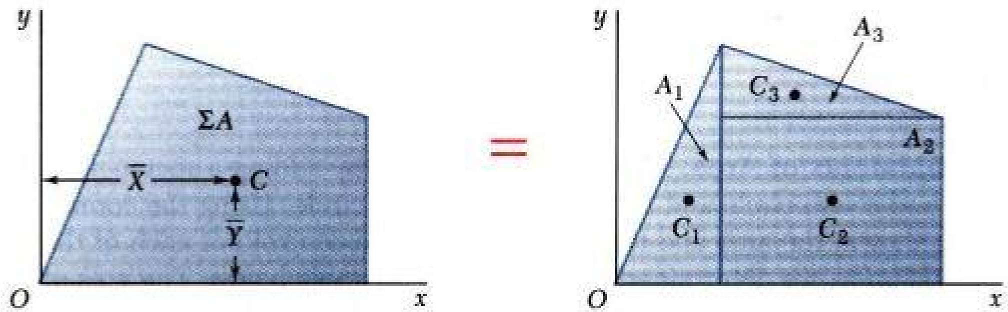
Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Composite Plates and Areas



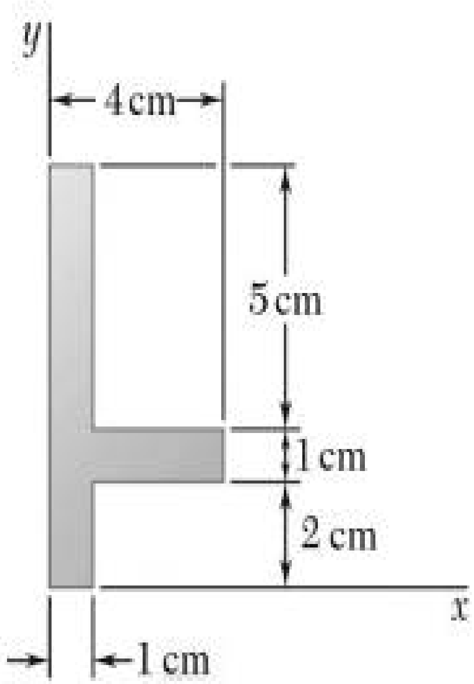
- Composite plates

$$\bar{X}\sum W = \sum \bar{x} W$$
$$\bar{Y}\sum W = \sum \bar{y} W$$



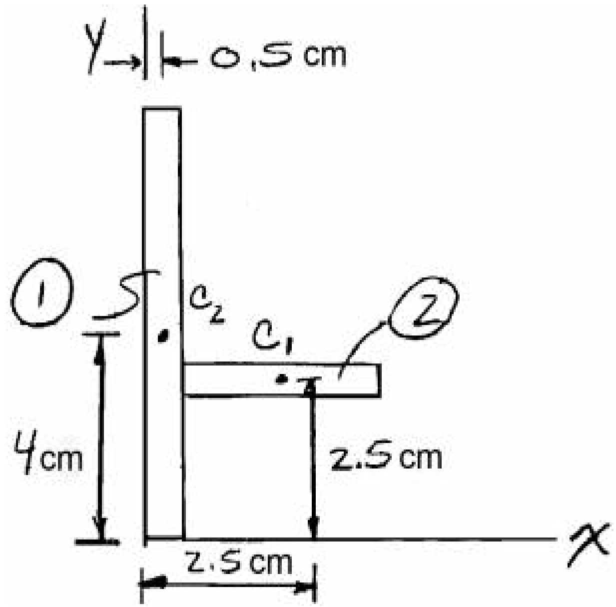
- Composite area

$$\bar{X}\sum A = \sum \bar{x} A$$
$$\bar{Y}\sum A = \sum \bar{y} A$$



PROBLEM 5.1

Locate the centroid of the plane area shown.



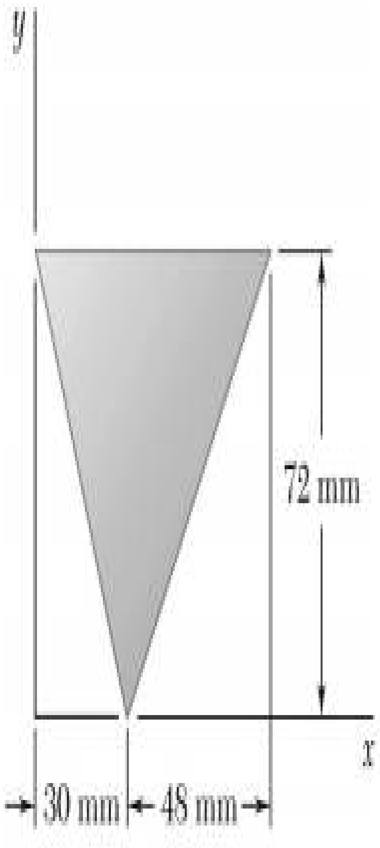
	A, cm^2	$\bar{x}$, cm	$\bar{y}$, cm	$\bar{x}A, cm^3$	$\bar{y}A, cm^3$
1	8	0.5	4	4	32
2	3	2.5	2.5	7.5	7.5
Σ	11			11.5	39.5

$$\bar{X}\Sigma A = \bar{x}A$$

$$\bar{X}(11\,\text{cm}^2) = 11.5\,\text{cm}^3 \qquad \bar{X} = 1.045\,\text{cm.} \blacktriangleleft$$

$$\bar{Y}\Sigma A = \Sigma\bar{y}A$$

$$\bar{Y}(11) = 39.5 \qquad \bar{Y} = 3.591\,\text{cm.} \blacktriangleleft$$

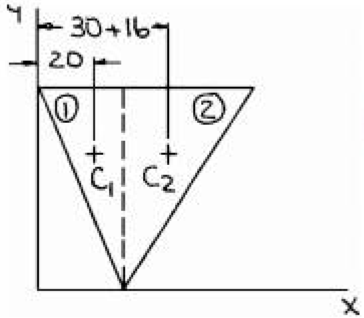


PROBLEM 5.2

Locate the centroid of the plane area shown.

$$\bar{Y} = \frac{2}{3}(72 \text{ mm})$$

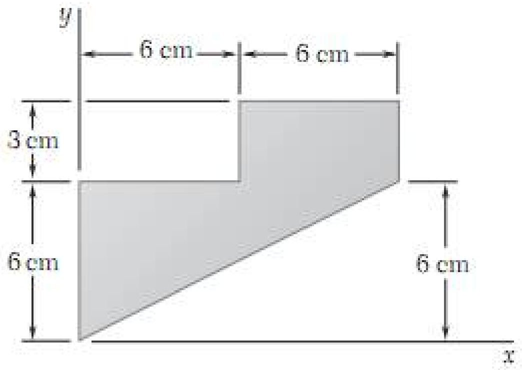
or $\bar{Y} = 48.0$ mm ◀



Dimensions in mm

	A, mm^2	$\bar{x}$, mm	$\bar{x}A$, mm^3
1	$\frac{1}{2} \times 30 \times 72 = 1080$	20	21,600
2	$\frac{1}{2} \times 48 \times 72 = 1728$	46	79,488
Σ	2808		101,088

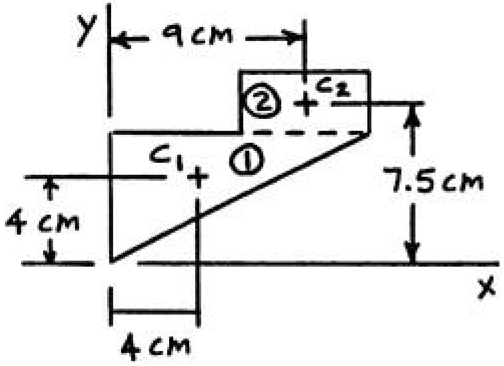
Then $\bar{X}A = \Sigma \bar{x}A$ $\bar{X}(2808) = 101{,}088$ or $\bar{X} = 36.0$ mm ◀



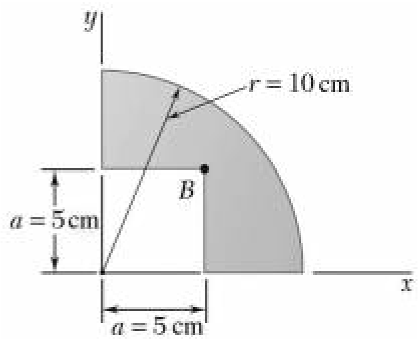
PROBLEM 5.4

Locate the centroid of the plane area shown.

SOLUTION



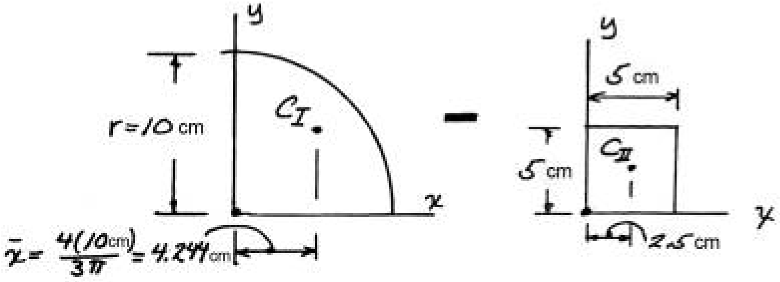
	A, cm^2	$\overline{x}$, cm	$\overline{y}$, cm	$\overline{x}A$, cm^3	$\overline{y}A$, cm^3
1	$\frac{1}{2}(12)(6) = 36$	4	4	144	144
2	$(6)(3) = 18$	9	7.5	162	135
Σ	54			306	279



PROBLEM 5.5

Locate the centroid of the plane area shown.

SOLUTION



By symmetry, $\bar{X} = \bar{Y}$

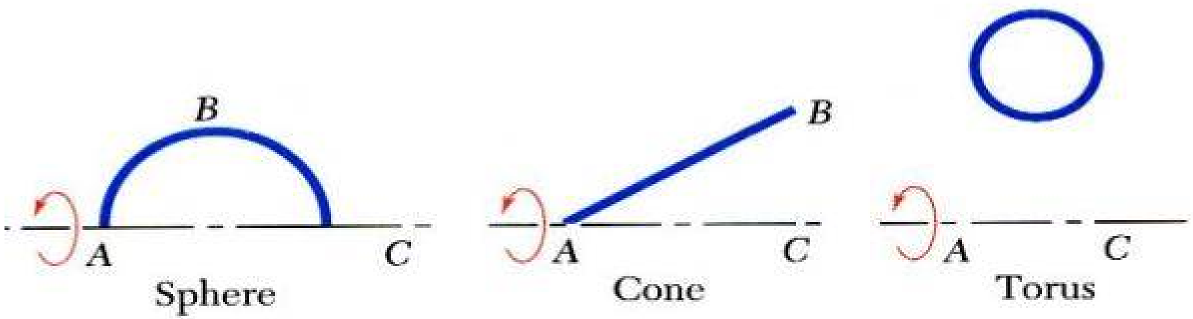
	Component	A, cm^2	$\bar{x}$, cm	$\bar{x}A$, cm^3
I	Quarter circle	$\frac{\pi}{4}(10)^2 = 78.54$	4.244	333.32
II	Square	$-(5)^2 = -25$	2.5	-62.5
Σ		53.54		270.82

$$\bar{X}\Sigma A = \Sigma\bar{x}A: \quad \bar{X}(53.54\text{ cm}^2) = 270.82\text{ cm}^3$$

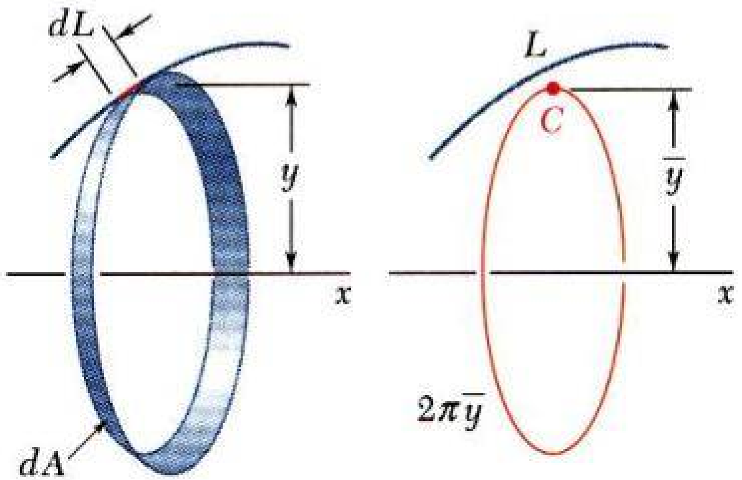
$$\bar{X} = 5.0583\text{ cm}$$

$$\bar{X} = \bar{Y} = 5.058\text{ cm}$$ ◀

Theorems of Pappus-Guldinus



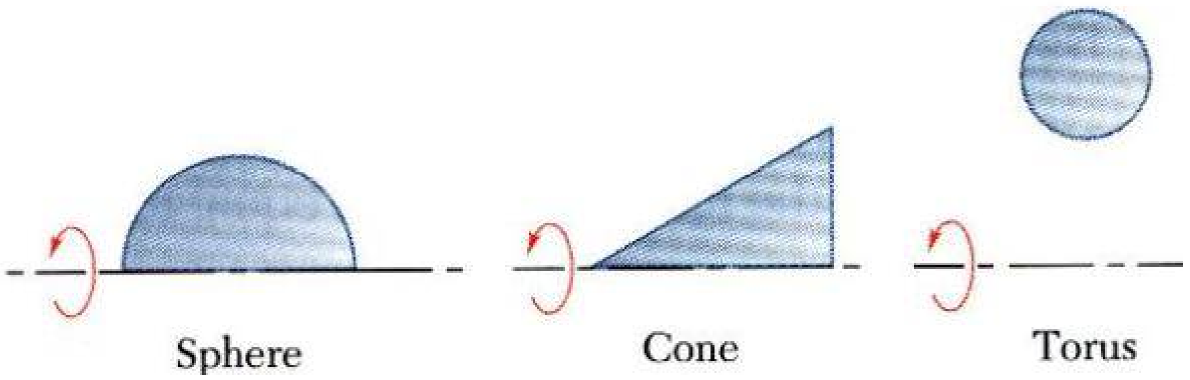
- Surface of revolution is generated by rotating a plane curve about a fixed axis.



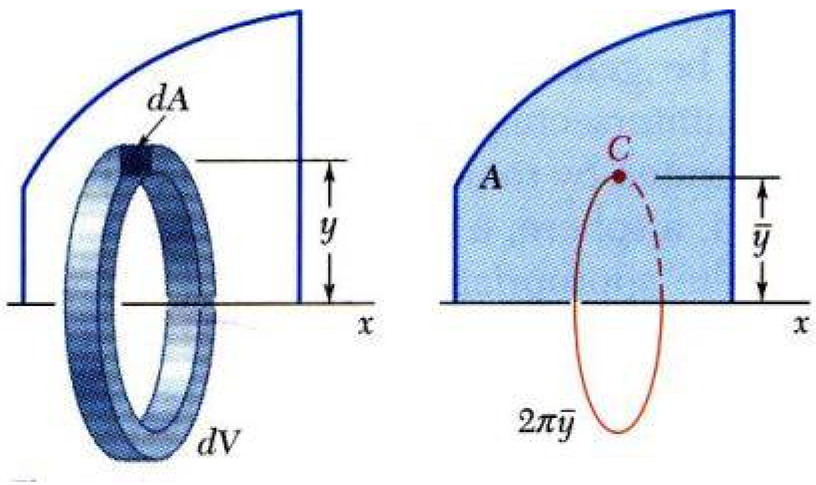
- **Theorem I**: Area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid through the rotation.

$$A = 2\pi \bar{y} L$$

Theorems of Pappus Guldinus

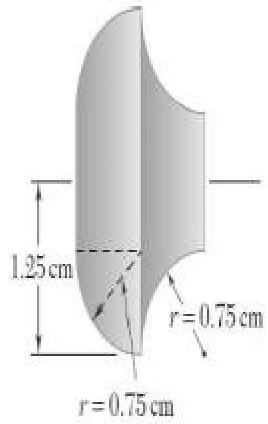


- Body of revolution is generated by rotating a plane area about a fixed axis.



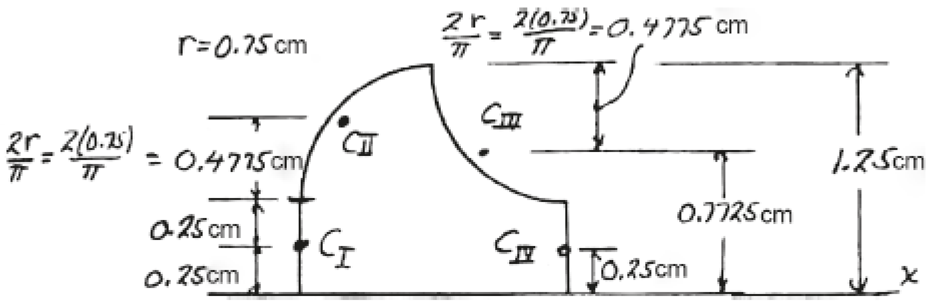
- **Theorem II**: Volume of a body of revolution is equal to the generating area times the distance traveled by the centroid through the rotation.

$$V = 2\pi \bar{y} A$$



PROBLEM 5.59

Determine the total surface area of the solid brass knob shown.



Area is obtained by rotating lines shown about the x-axis.

	L, cm	$\bar{y}$, cm	$\bar{y}L$, cm^2
1	0.5	0.25	0.1250
2	$\frac{\pi}{2}(0.75) = 1.1781$	0.9775	1.1516
3	$\frac{\pi}{2}(0.75) = 1.1781$	0.7725	0.9101
4	0.5	0.25	0.1250
Σ			2.3117

$$A = 2\pi \Sigma \bar{y} L = 2\pi (2.3117 \text{ cm}^2)$$

References:

1. Beer, Ferdinand P.; Johnston, E. Russell; "Vector Mechanics for Engineers - Statics", 8th Ed., McGraw-Hill, Singapore, 2007.