UMP OPEN COURSEWARE

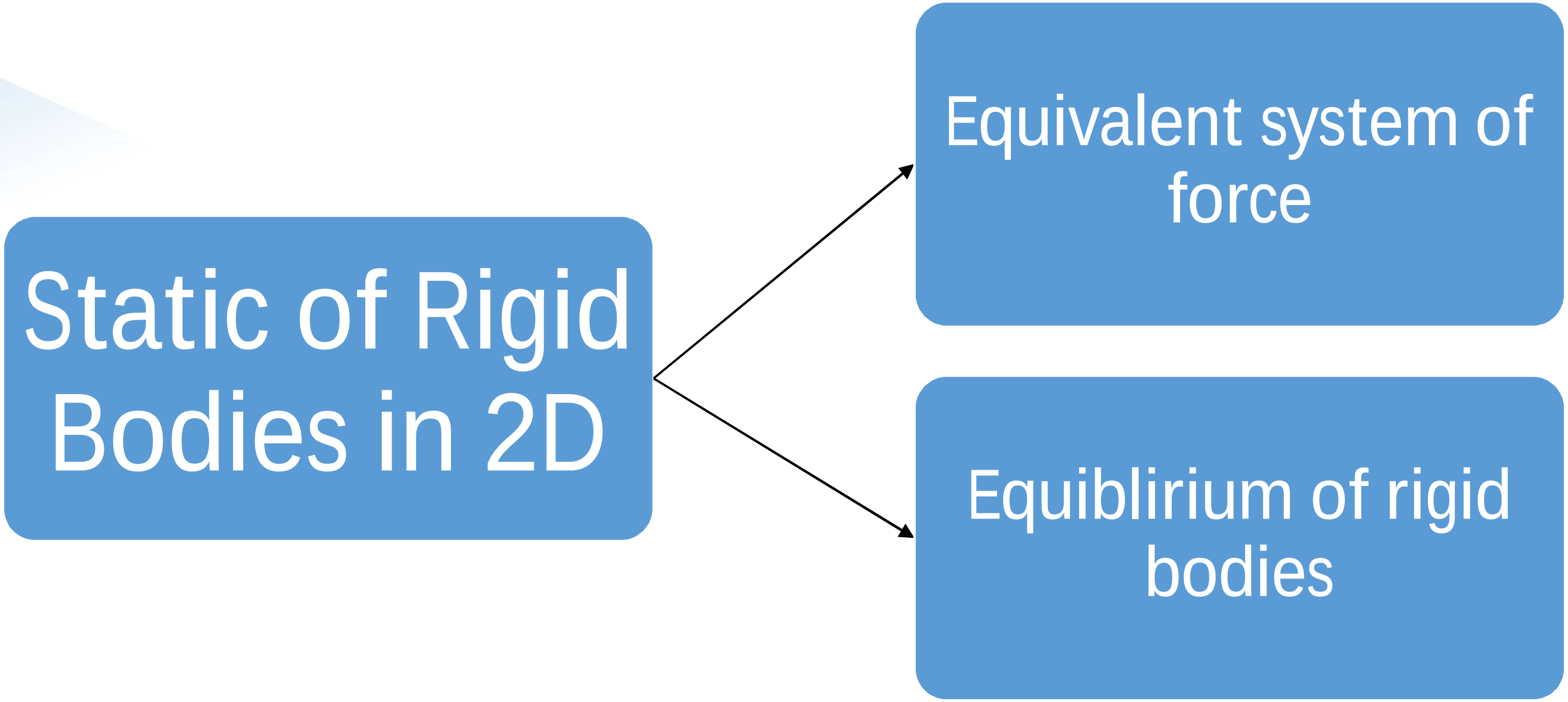
Universiti Malaysia PAHANG
Engineering • Technology • Creativity

CHAPTER 3

STATICS OF RIGID BODIES IN TWO DIMENSION

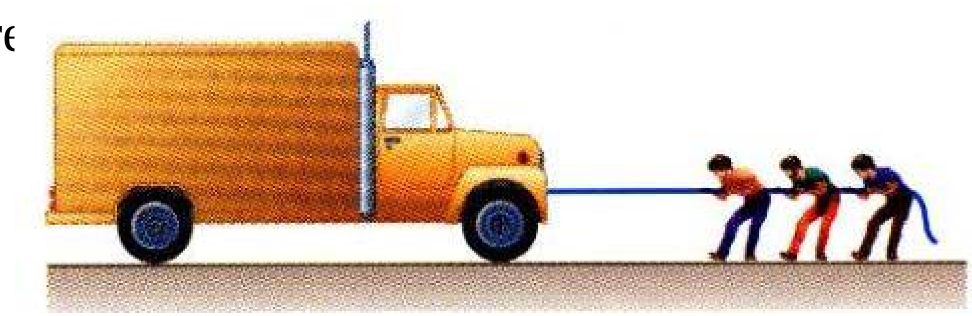
Expected Outcome:

- Able to identify all external forces and their directions, acting on a rigid body
- Able to calculate the moment of a force about a point
- Able to analyze and replace a given force acting on a rigid body with an equivalent system of forces
- Draw a free body diagram for a rigid body and solve problems involving the equilibrium of a rigid body using the three equations of equilibrium or, if possible, using the concept of equilibrium of a 3-Force Body

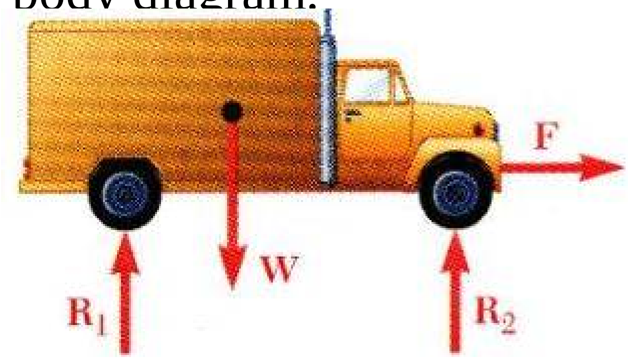

Static of Rigid Bodies in 2D
Equivalent system of force
Equiblirium of rigid bodies

External and Internal Forces

- Forces acting on rigid bodies are
  - External forces
  - Internal forces

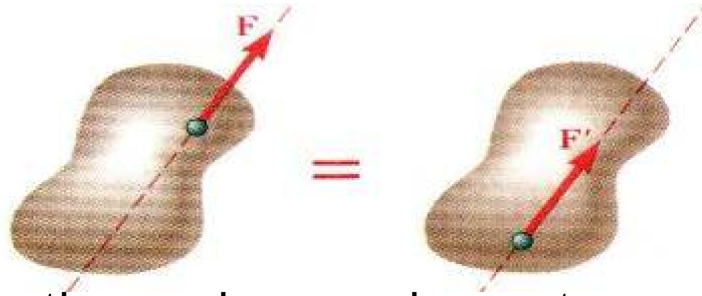


- External forces are shown in a free-body diagram.

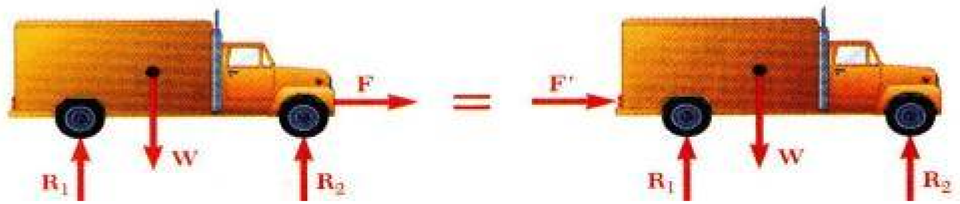


Fundamental concept in Equivalent force:

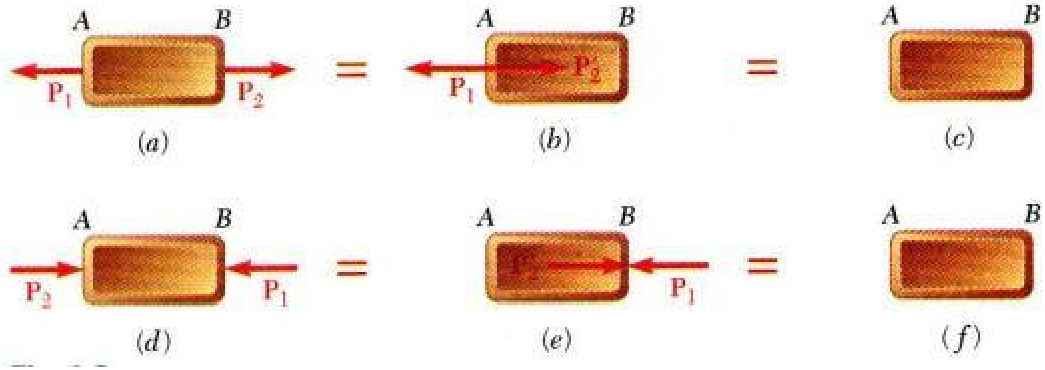
- *Principle of Transmissibility* -
  Conditions of equilibrium or motion are not affected by *transmitting* a force along its line of action.
  NOTE: **F** and **F'** are equivalent forces.



- Moving the point of application of the force **F** to the rear bumper does not affect the motion or the other forces acting on the truck.



- Principle of transmissibility may not always apply in determining internal forces and deformations.



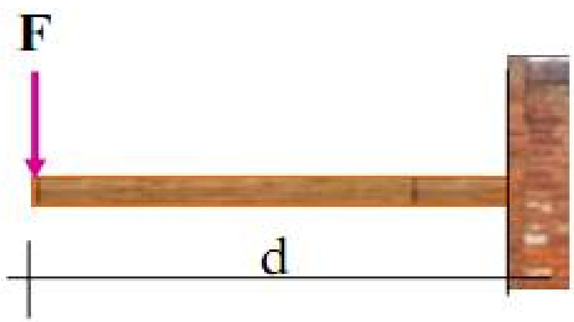
- Concept of Moment

What is moment?

Moment of a Force

This tendency of a force to produce rotation about some point is called the Moment of a force

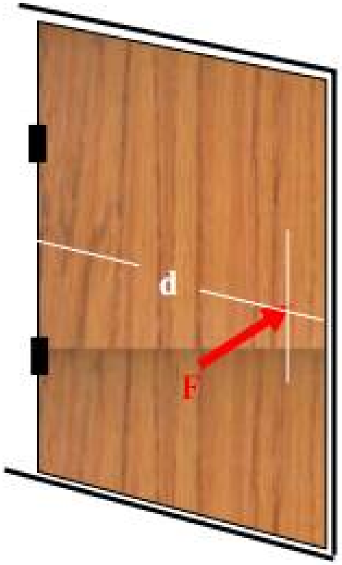
Moment of a Force



The tendency of a force to produce rotation of a body about some reference axis or point is called the MOMENT OF A FORCE

$$M=Fxd$$

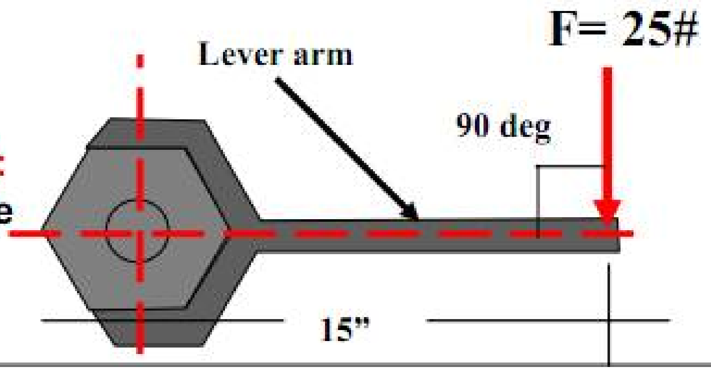
Common Examples in the Application of the Concept of Moment



Example One: Closing the Door

Moment = Force x Perpendicular Distance = Fxd

Example Two: Tightening the NUT

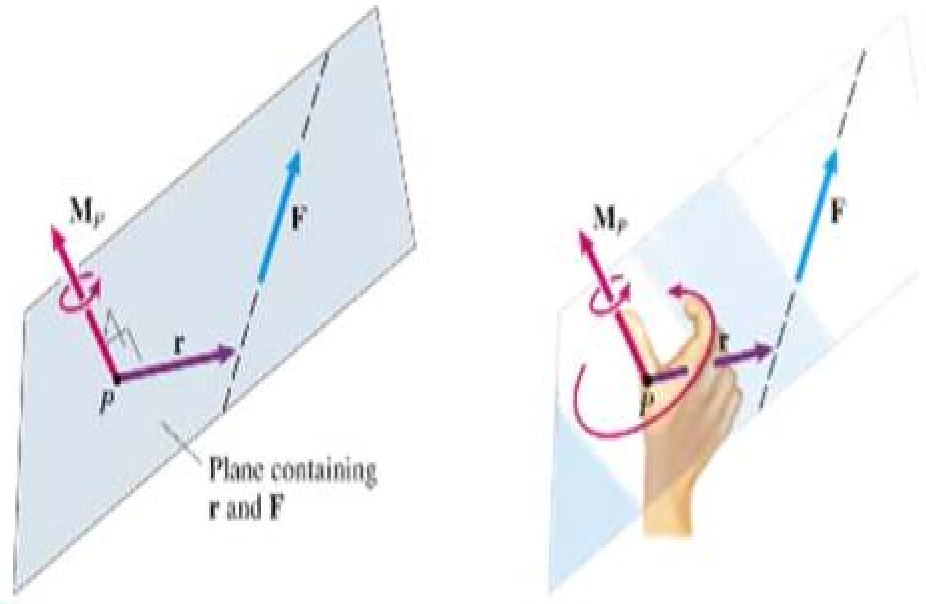


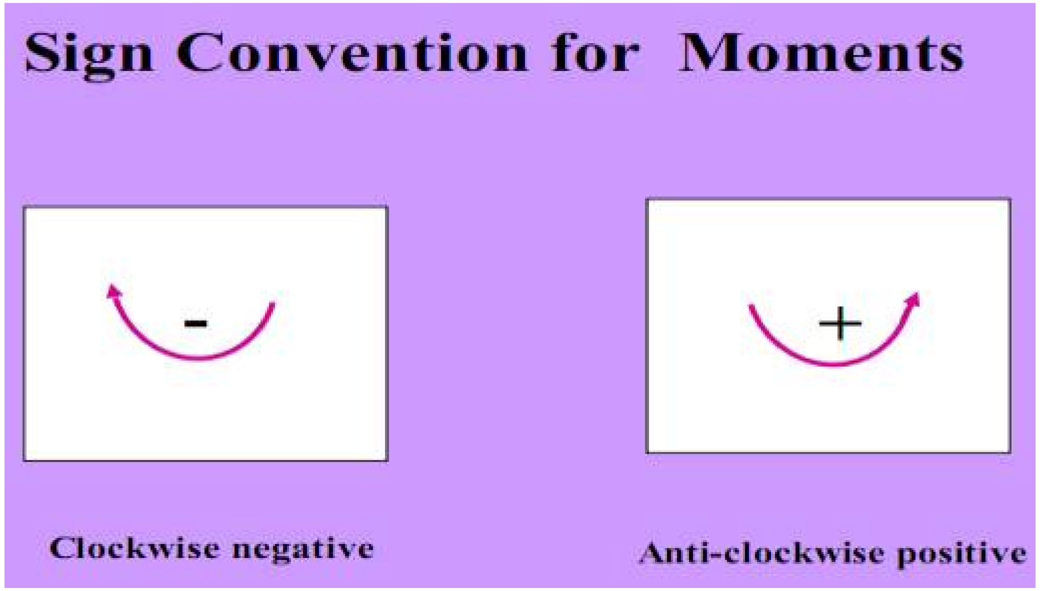
$M = - F \times d$

$= -25 \times 15$

$= - 375$ #-in

Direction of the Moment:



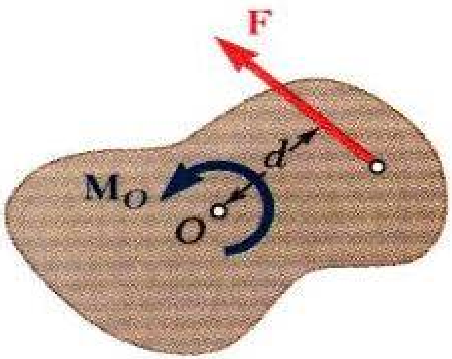


Moment of a Force About a Point

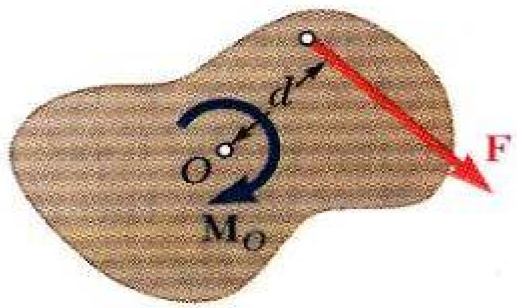
- *Condition/ situation occur?*

When Two-dimensional structures have length and breadth but negligible depth and are subjected to forces contained in the plane of the structure.

- The plane of the structure contains the point *O* and the force ***F***. $\boldsymbol{M_O}$, the moment of the force about *O* is perpendicular to the plane.
- If the force tends to rotate the structure clockwise, the sense of the moment vector is into the plane of the structure and the magnitude of the moment is negative.
- If the force tends to rotate the structure counterclockwise, the sense of the moment vector is out of the plane of the structure and the magnitude of the moment is positive.

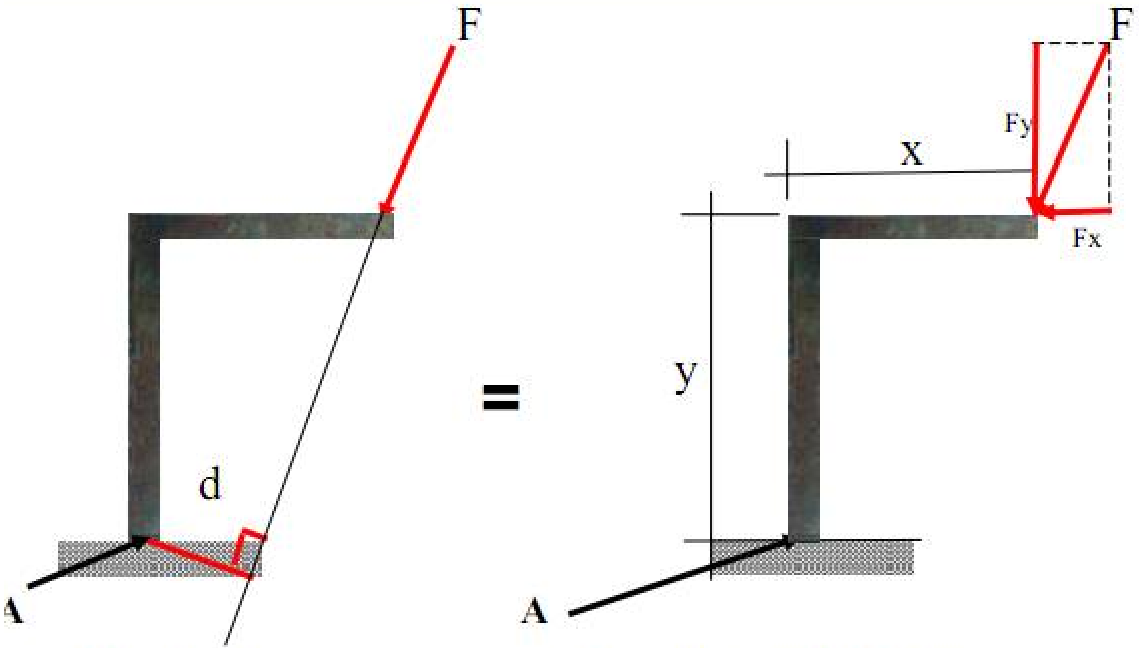


(a) $\mathbf{M}_O = +Fd$



(b) $\mathbf{M}_O = -Fd$

Varignon's Theorem



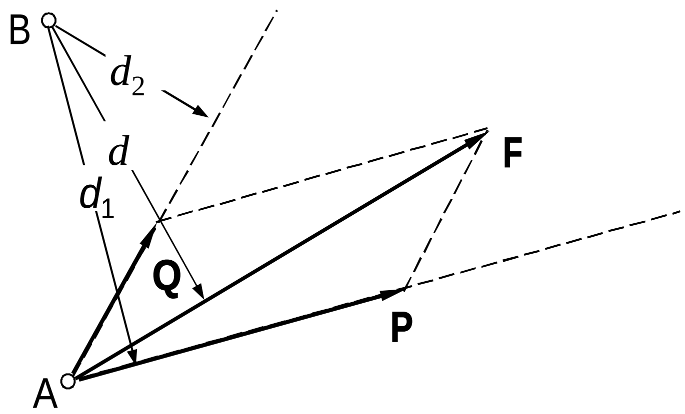
M=-F.d

M= -Fy.x + Fx.y

According to Varignon's Theorem, a Force can be resolved into its components and multiplied by the perpendicular distances for easy calculation of the Moment

Varignon’s Theorem

•The moment of a force about any axis is equal to the sum of the moments of its components about that axis.

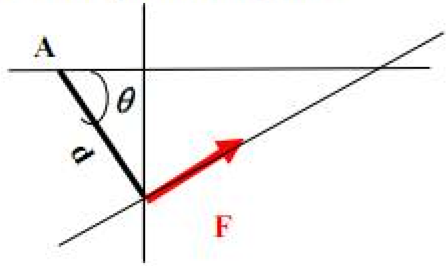


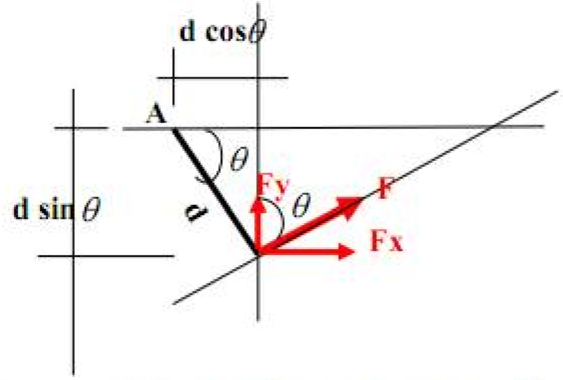
Varignon’s Theorem states that:

$$Fd = Pd_1 + Qd_2$$

“PROOF OF VARIGON’S THEOREM”

Proof of Varignon’s Theorem





M about A= F x d

$$F(d) = F_y(d\cos\theta) + F_x(d\sin\theta)$$

Substitute for Fx and Fy

$$F(d) = F\cos\theta(d\cos\theta) + F\sin\theta(d\sin\theta)$$

$$F(d) = Fd\cos^2\theta + Fd\sin^2\theta$$

$$Fd = Fd(\cos^2\theta + \sin^2\theta)$$

F(d)=F x d

PYTHAGOREAN IDENTITIES

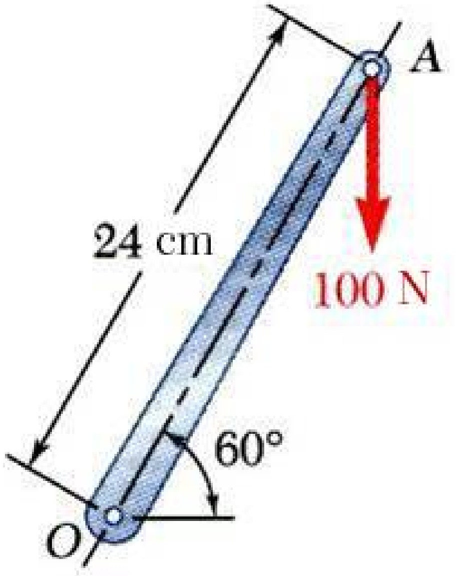
$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

F x d = ***Fd***

Sample Problem 3.1



A 100-N vertical force is applied to the end of a lever which is attached to a shaft at *O*.

Determine:

a) moment about *O*,

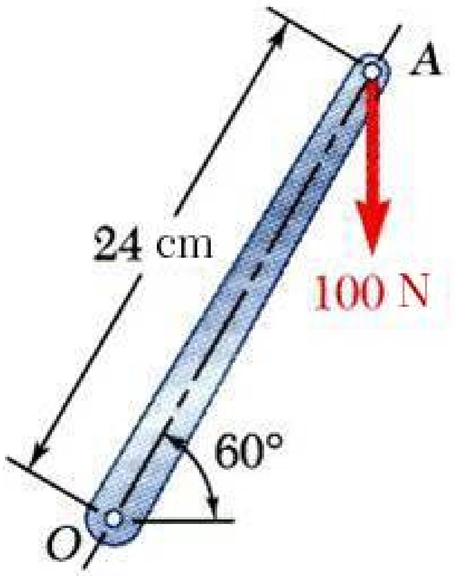
b) horizontal force at *A* which creates the same moment,

c) smallest force at A which produces the same moment,

d) location for a 240-N vertical force to produce the same moment,

e) whether any of the forces from b, c, and d is equivalent to the original force.

Sample Problem 3.1

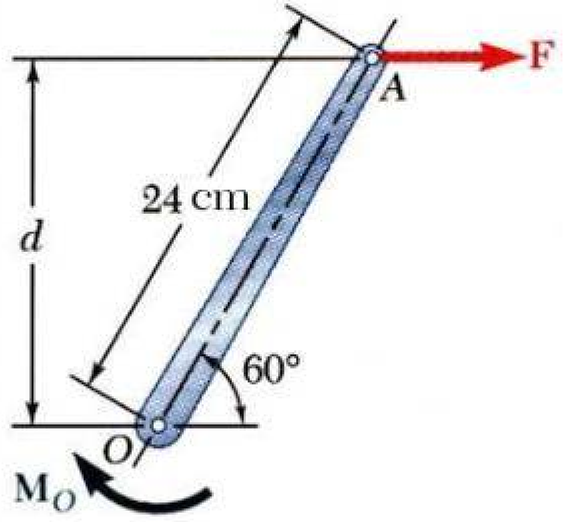


a) $M_O = Fd$

$$d = (24\,\text{cm})\cos 60° = 12\,\text{cm}$$

$$M_O = (100\,\text{N})(12\,\text{cm})$$

$$M_O = 1.2\ \text{N}\cdot\text{m}$$



b) Horizontal force at *A* that produces the same moment,

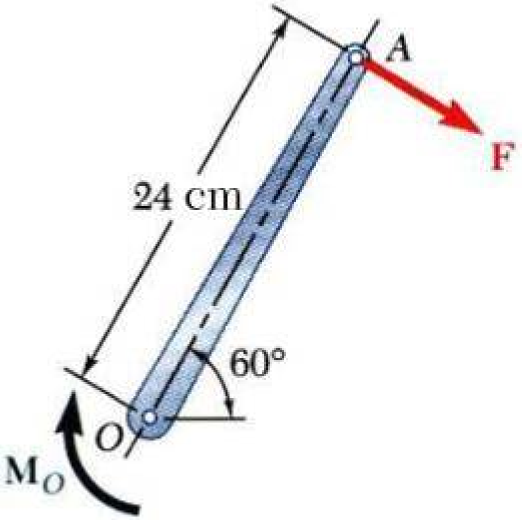
$$d = (24\text{ cm})\sin 60° = 20.8\text{ cm}$$

$$M_O = Fd$$

$$1200\text{ N}\cdot\text{cm} = F(20.8\text{ cm})$$

$$F = \frac{1200\text{ N}\cdot\text{cm}}{20.8\text{ cm}}$$

$$\boxed{F = 57.7\text{ N}}$$



c) The smallest force *A* to produce the same moment occurs when the perpendicular distance is a maximum or when *F* is perpendicular to *OA*.

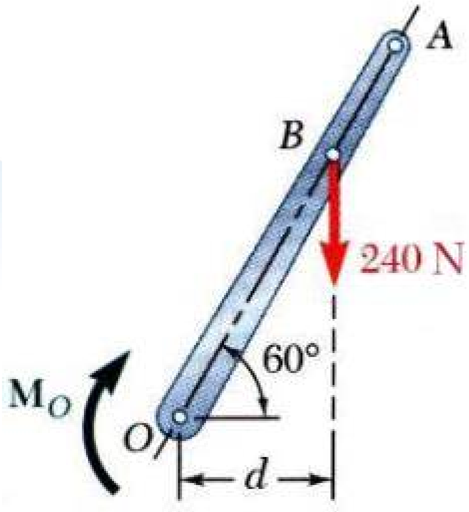
$$M_O = Fd$$

$$1200\ \text{N}\cdot\text{cm} = F(24\ \text{cm})$$

$$F = \frac{1200\ \text{N}\cdot\text{cm}}{24\ \text{cm}}$$

$$\boxed{F = 50\ \text{N}}$$

d) To determine the point of application of a 240 N force to produce the same moment,



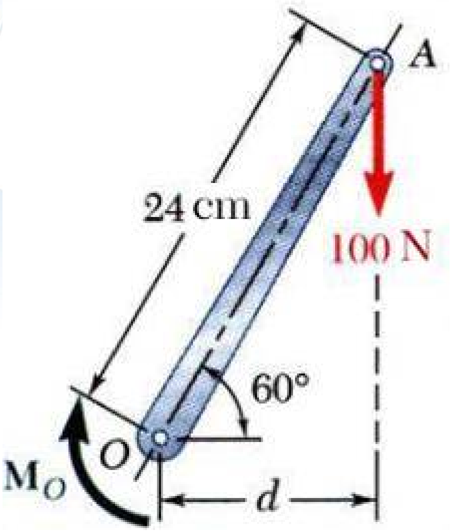
$$M_O = Fd$$

$$1200\ \text{N}\cdot\text{cm} = (240\ \text{N})d$$

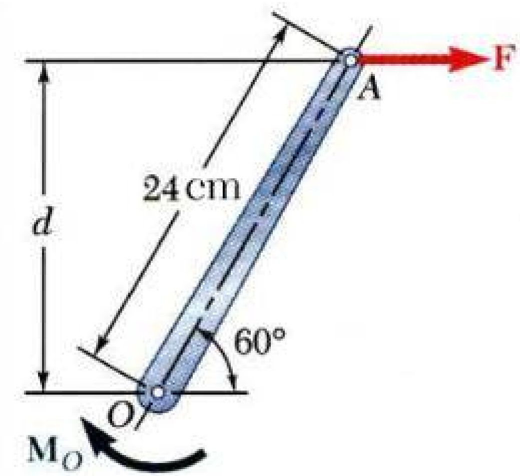
$$d = \frac{1200\ \text{N}\cdot\text{cm}}{240\ \text{N}} = 5\ \text{cm}$$

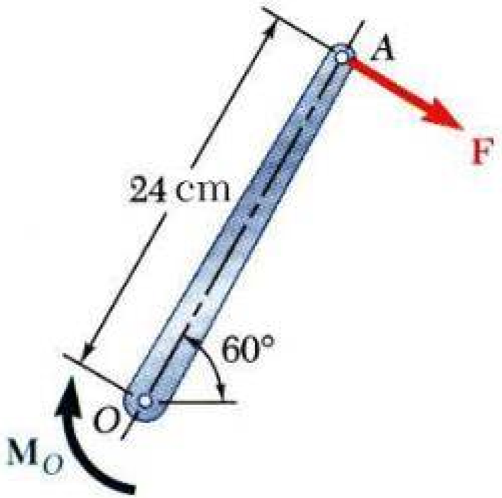
$$OB\cos 60° = 5\ \text{cm}$$

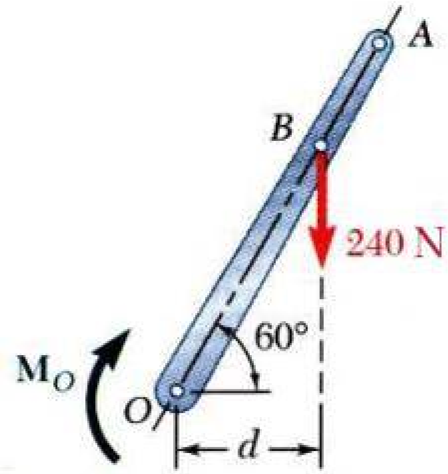
$$\boxed{OB = 10\ \text{cm}}$$



e) Although each of the forces in parts b), c), and d) produces the same moment as the 100 N force, none are of the same magnitude and sense, or on the same line of action. None of the forces is equivalent to the 100 N force.

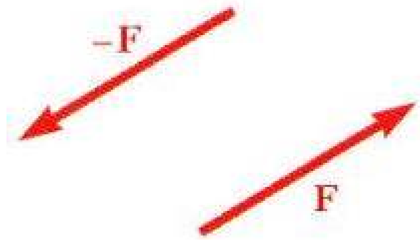






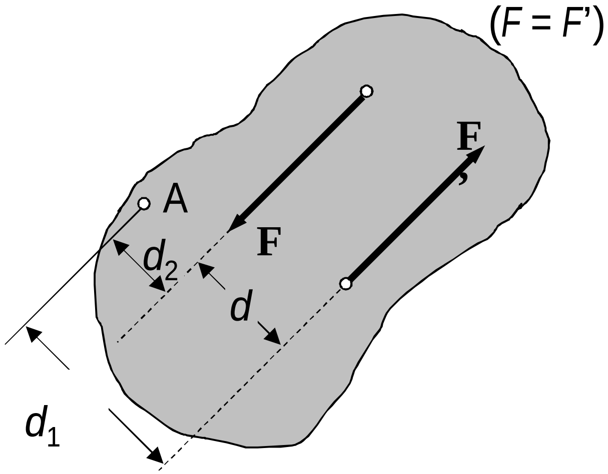
Moment of a Couple

- Two forces ***F*** and ***-F*** having the same magnitude, parallel lines of action, and opposite sense are said to form a *couple*.



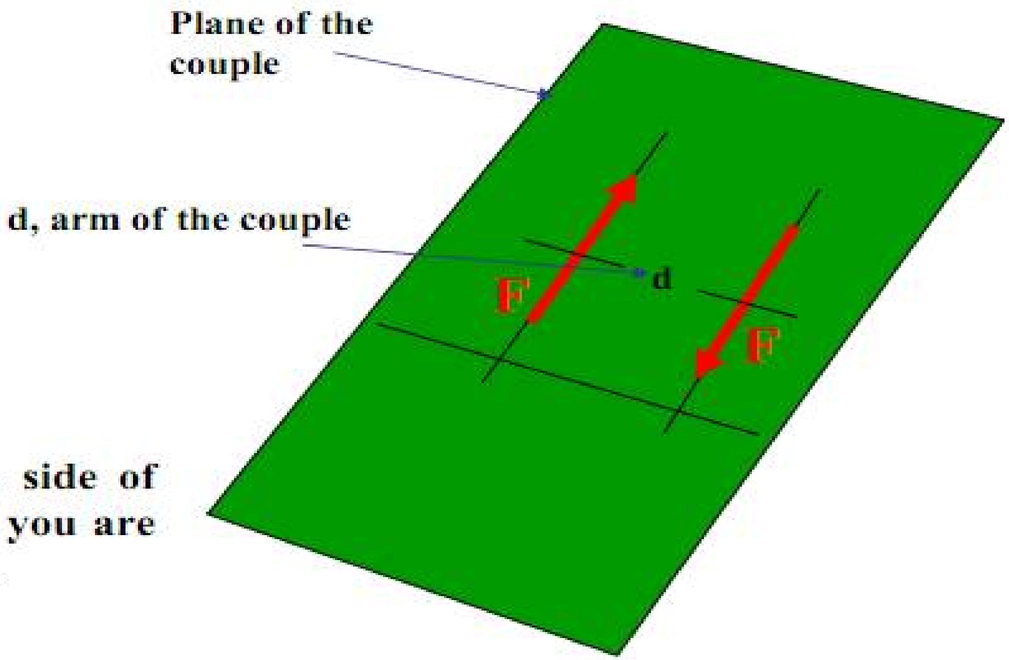
- Moment of a couple,

$$
\begin{aligned}
+ve \circlearrowleft \; M &= Fd_1 - Fd_2 \\
&= F(d_1 - d_2) \\
&= Fd
\end{aligned}
$$



- The moment M of a couple is constant. Its magnitude is equal to the product Fd of their common magnitude F and the distance between their lines of action. **The sense of M (clockwise or counterclockwise) is obtained by direct observation**.

Concept of a Couple

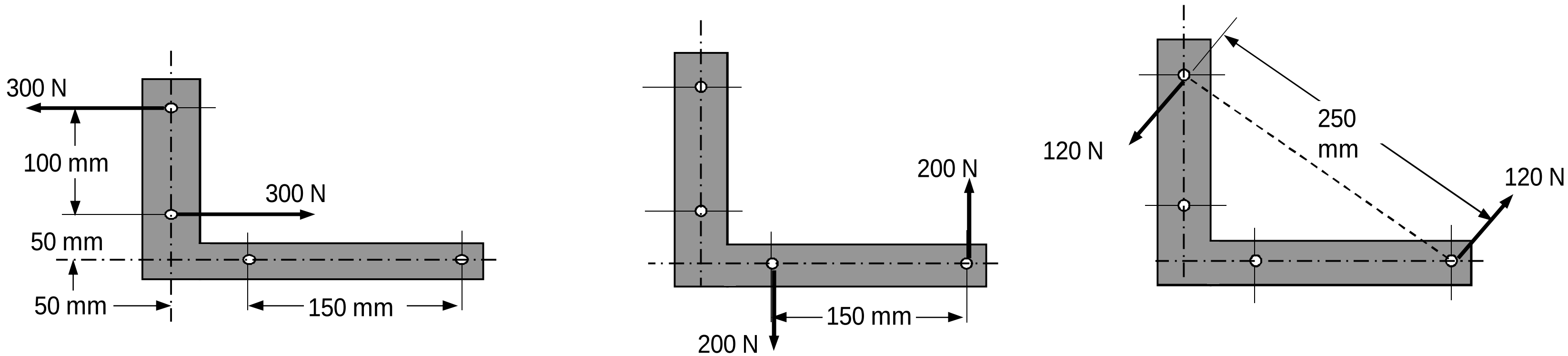


When you grasp the opposite side of the steering wheel and turn it, you are applying a couple to the wheel.

A couple is defined as two forces (coplanar) having the same magnitude, parallel lines of action, but opposite sense. Couples have pure rotational effects on the body with no capacity to translate the body in the vertical or horizontal direction. (Because the sum of their horizontal and vertical components are zero)

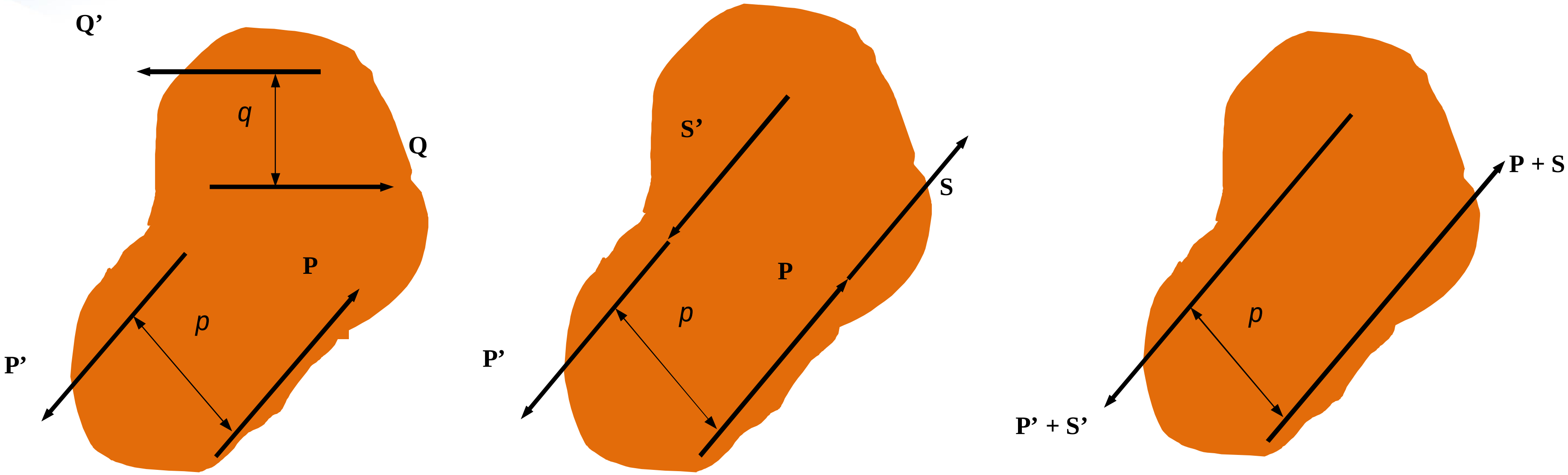
Equivalent Couples

- Couples having the same moment, both in magnitude and sense, are equivalent.
- Each will have the same effect on a rigid body.
- When a couple acts on a rigid body, **it does not matter where the two forces forming the couple act, or what magnitude and direction they have.**
- The **only thing that counts is the moment (magnitude and sense) of the couple**.



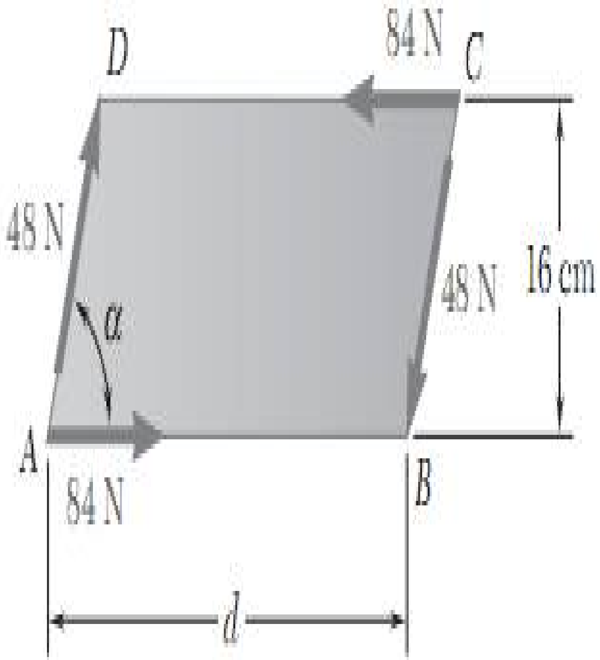
Addition of Couples

- Two couples formed by **P**, **P’** and **Q**, **Q’** acting on a rigid body may be replaced by a single couple of moment equal to the algebraic sum of the moments of the given couples.



$$Sp = Qq$$

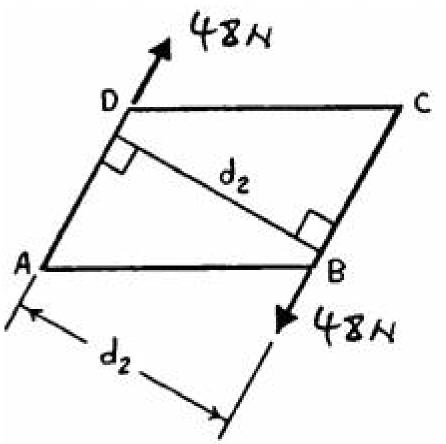
$$M = (P+S)p = Pp + Sp = Pp + Qq$$



PROBLEM 3.70

A plate in the shape of a parallelogram is acted upon by two couples. Determine (*a*) the moment of the couple formed by the two 84 N forces, (*b*) the perpendicular distance between the 48 N forces if the resultant of the two couples is zero, (*c*) the value of α if the resultant couple is 2.88 N·m clockwise and d is 42 cm.

SOLUTION



(a) We have $M_1 = d_1 F_1$

where $d_1 = 16 \text{ cm} = 0.16 \text{ m}$

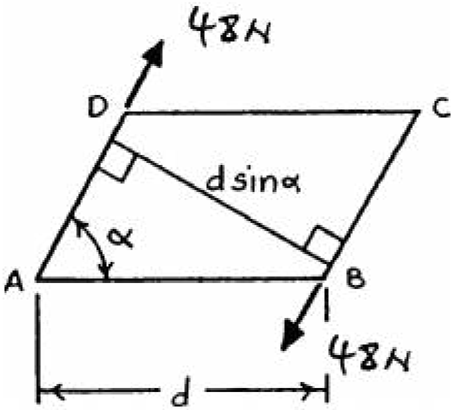
$F_1 = 84 \text{ N}$

$M_1 = (0.16 \text{ m})(84 \text{ N})$

$= 13.44 \text{ N·m}$ or $\mathbf{M}_1 = 13.4 \text{ N·m}$ ↻ ◀

(b) We have $\mathbf{M}_1 + \mathbf{M}_2 = 0$

or $13.44 \text{ N·m} - d_2(48 \text{ N}) = 0$ $d_2 = 0.28 \text{ m}$ ◀



(c) We have $\mathbf{M}_{total} = \mathbf{M}_1 + \mathbf{M}_2$

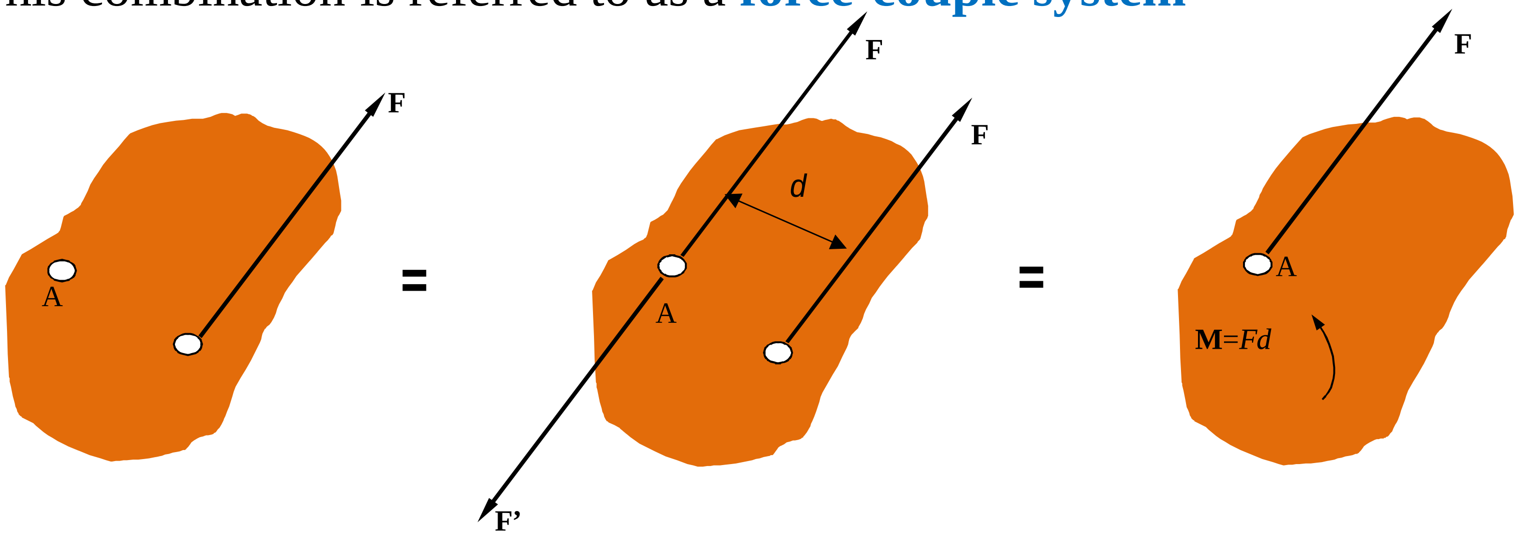
or $-2.88 \text{ N·m} = 13.44 \text{ N·m} - (0.42 \text{ m})(\sin\alpha)(48 \text{ N})$

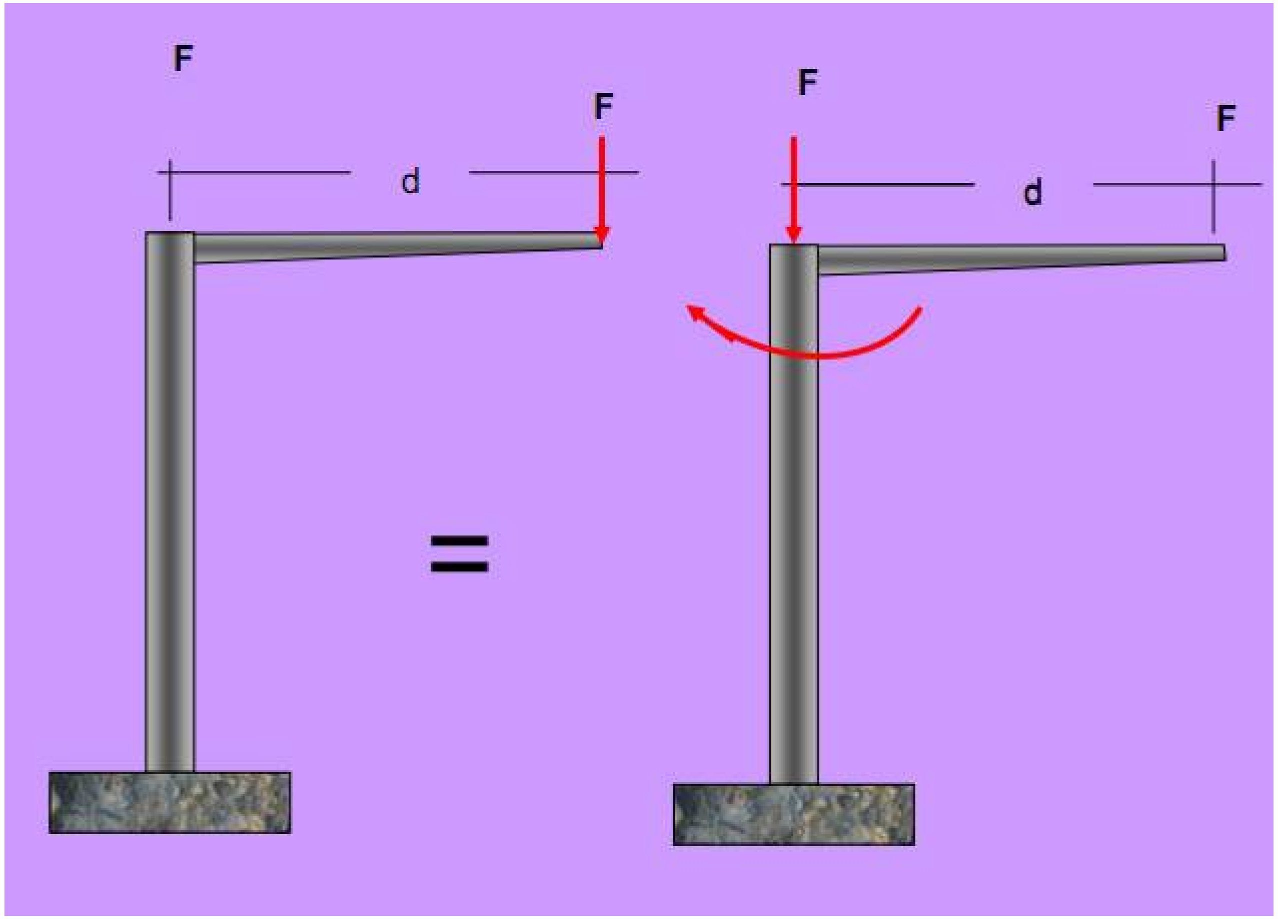
$\sin\alpha = 0.80952$

and $\alpha = 54.049°$ or $\alpha = 54.0°$ ◀

Resolution of a Given Force into a Force Acting at a Given Point and a Couple

- ANY FORCES F ACTING ON A RIGID BODY CAN BE MOVED TO AN ARBITARY POINT "O" PROVIDED THAT A COUPLE IS ADDED WHOSE MOMENT IS EQUAL TO THE MOMENT OF F ABOUT "O"
- This combination is referred to as a **force-couple system**




F
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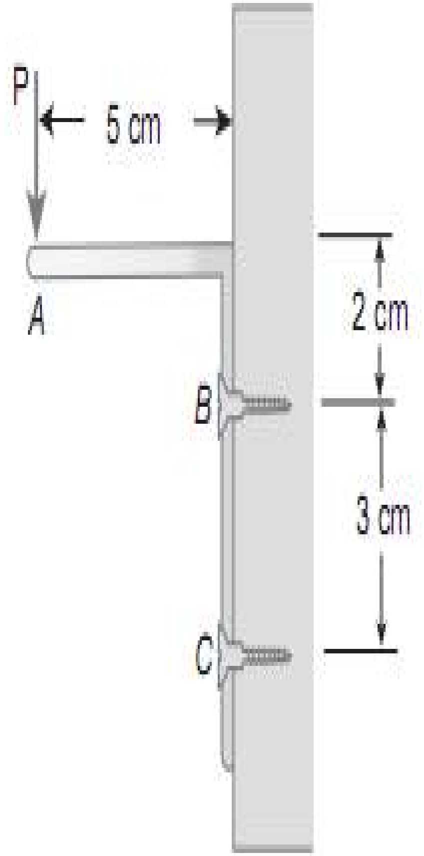
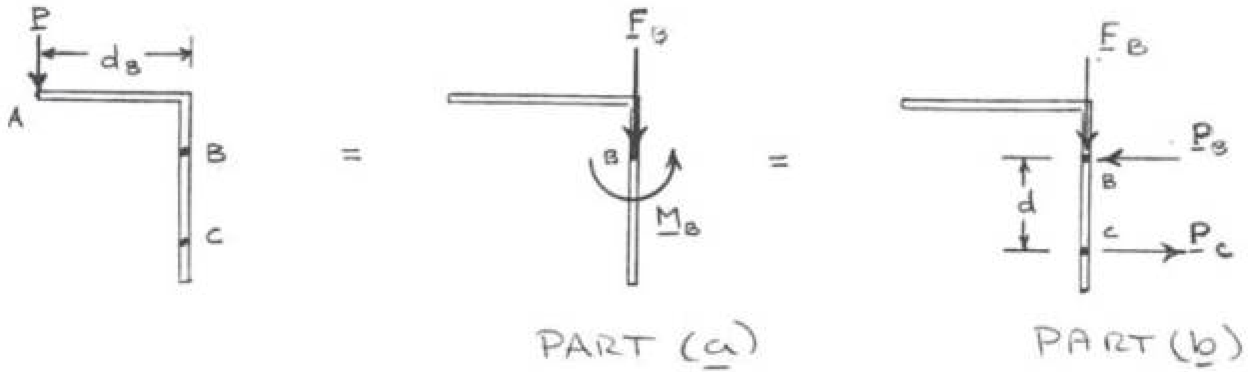


PROBLEM 3.82

A 30-N vertical force P is applied at *A* to the bracket shown, which is held by screws at *B* and *C*. (*a*) Replace **P** with an equivalent force couple system at *B*. (*b*) Find the two horizontal forces at *B* and *C* that are equivalent to the couple obtained in part *a*.

SOLUTION



(a) $\Sigma \mathbf{F}$: $\mathbf{F}_B = 30\text{ N}$

or $\mathbf{F}_B = 30\text{ N}\downarrow$ ◀

$\Sigma \mathbf{M}$: $M_B = P\, d_B$

$= (30\text{ N})\,(0.05\text{ m})$

$= 1.5\text{ N}\cdot\text{m}$

or $\mathbf{M}_B = 1.5\text{ N}\cdot\text{m}$ ↺ ◀

(*b*) $\Sigma \mathbf{M}_B$: $M_B = F_C d$

$$1.5 \text{ N} \cdot \text{m} = F_C (0.03 \text{ m})$$

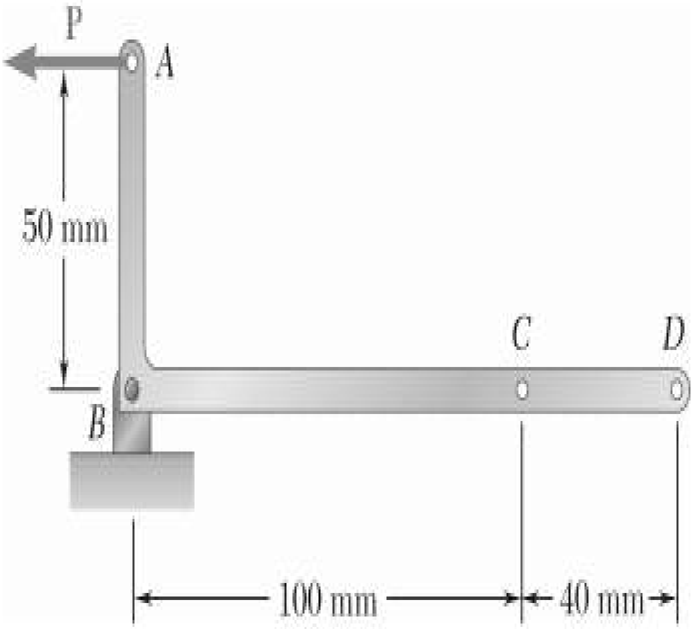
$$F_C = 50 \text{ N}$$

or $\mathbf{F}_C = 50 \text{ N} \longrightarrow$ ◀

$\Sigma \mathbf{F}$: $0 = -F_B + F_C$

$$F_B = F_C = 50 \text{ N}$$

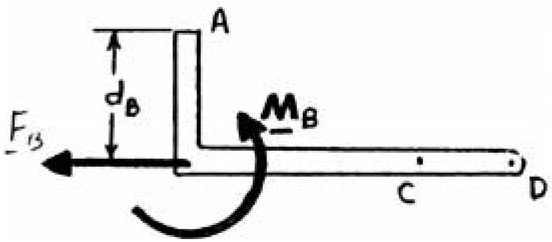
or $\mathbf{F}_B = 50 \text{ N} \longleftarrow$ ◀



PROBLEM 3.85

The 80-N horizontal force **P** acts on a bell crank as shown. (*a*) Replace **P** with an equivalent force-couple system at *B*. (*b*) Find the two vertical forces at *C* and *D* that are equivalent to the couple found in part *a*.

SOLUTION



(*a*) Based on $\Sigma F: \; F_B = F = 80 \text{ N}$ or $\mathbf{F}_B = 80.0 \text{ N} \longleftarrow$ ◀

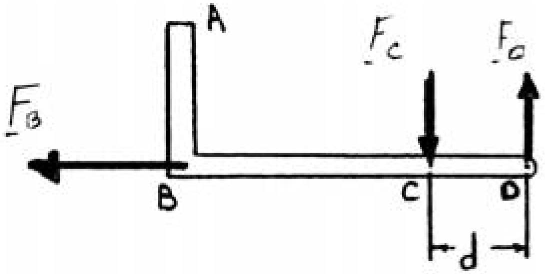
$\Sigma M: \; M_B = Fd_B$

$= 80 \text{ N } (0.05 \text{ m})$

$= 4.0000 \text{ N} \cdot \text{m}$

or $\mathbf{M}_B = 4.00 \text{ N} \cdot \text{m}$ ↺ ◀

(*b*) If the two vertical forces are to be equivalent to $\mathbf{M}_B$, they must be a couple. Further, the sense of the moment of this couple must be counterclockwise.



Then with F_C and F_D acting as shown,

$\Sigma M: \; M_D = F_C d$

$4.0000 \text{ N} \cdot \text{m} = F_C (0.04 \text{ m})$

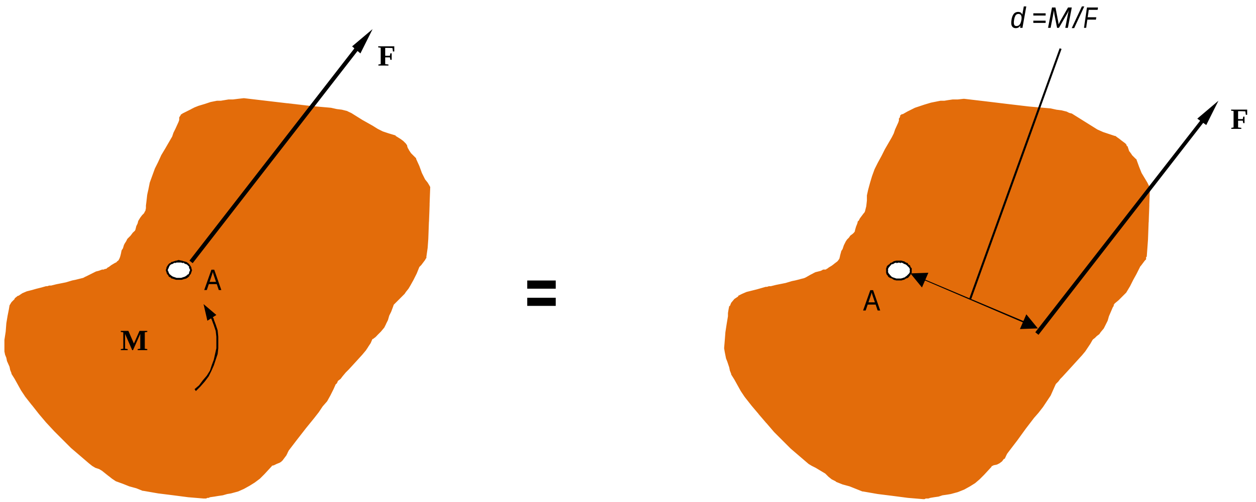
$F_C = 100.000 \text{ N}$ or $\mathbf{F}_C = 100.0 \text{ N} \downarrow$ ◀

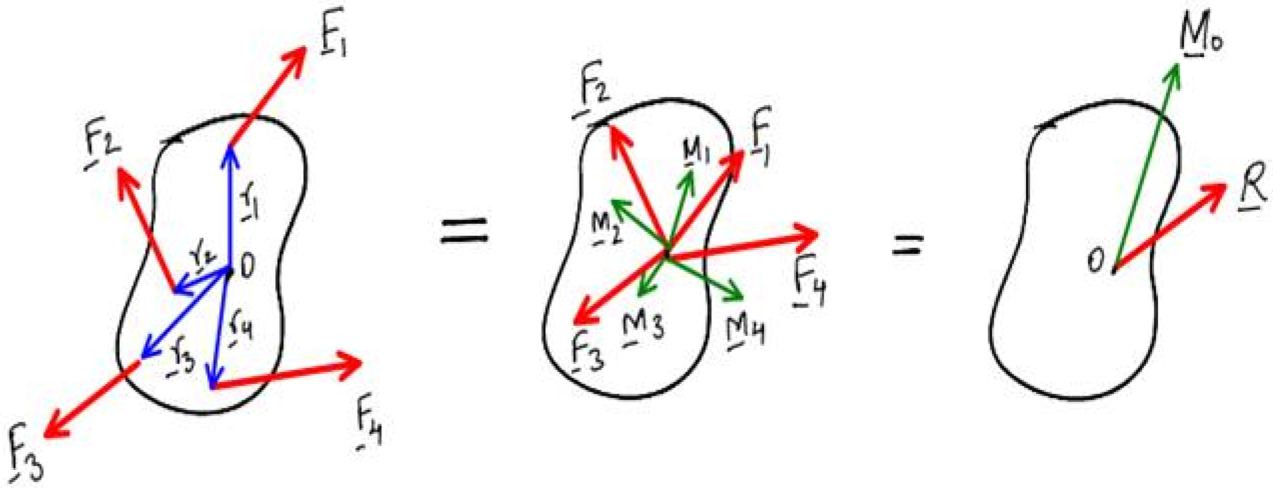
$\Sigma F_y: \; 0 = F_D - F_C$

$F_D = 100.000 \text{ N}$ or $\mathbf{F}_D = 100.0 \text{ N} \uparrow$ ◀

Reduction of a System of Coplanar Forces to One Force and One Couple. Resultant of a System of Coplanar Forces.

- Inversely, a force **F** acting at A and a couple **M** may be combined into a single resultant force **F**, by moving **F** such that the moment *M* of the couple is eliminated.





$$\underline{M}_1 = \underline{r}_1 \times \underline{F}_1$$

$$\underline{M}_2 = \underline{r}_2 \times \underline{F}_2$$

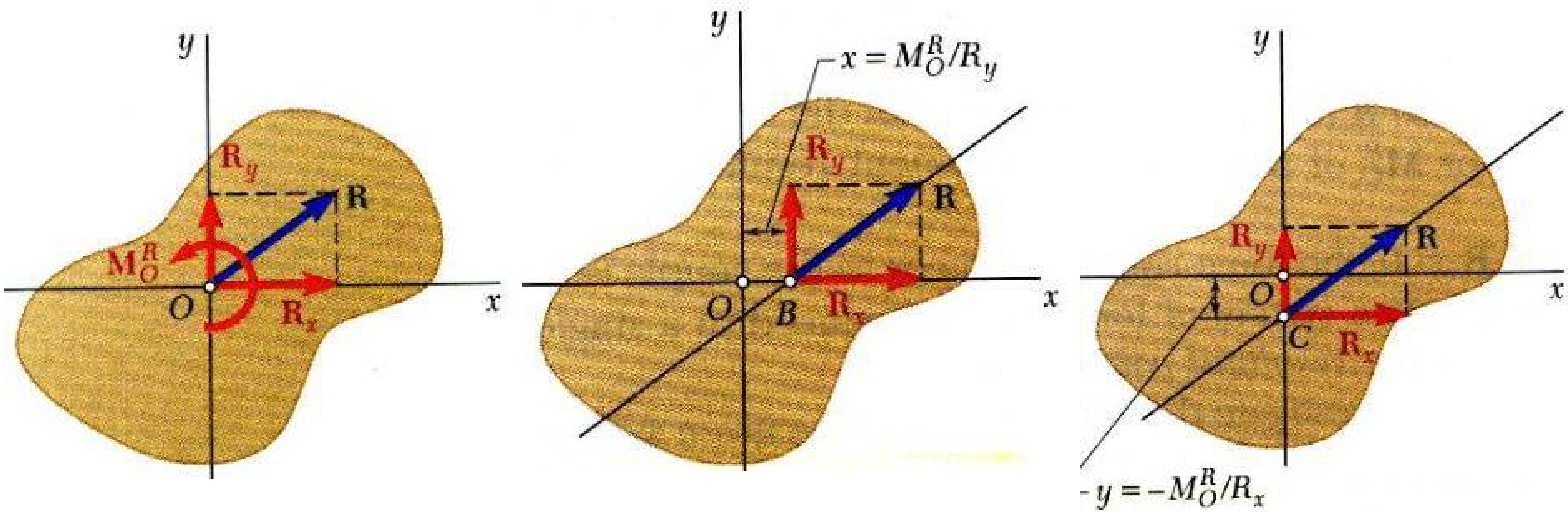
$$\underline{M}_3 = \underline{r}_3 \times \underline{F}_3$$

$$\vdots \qquad \vdots \qquad \vdots$$

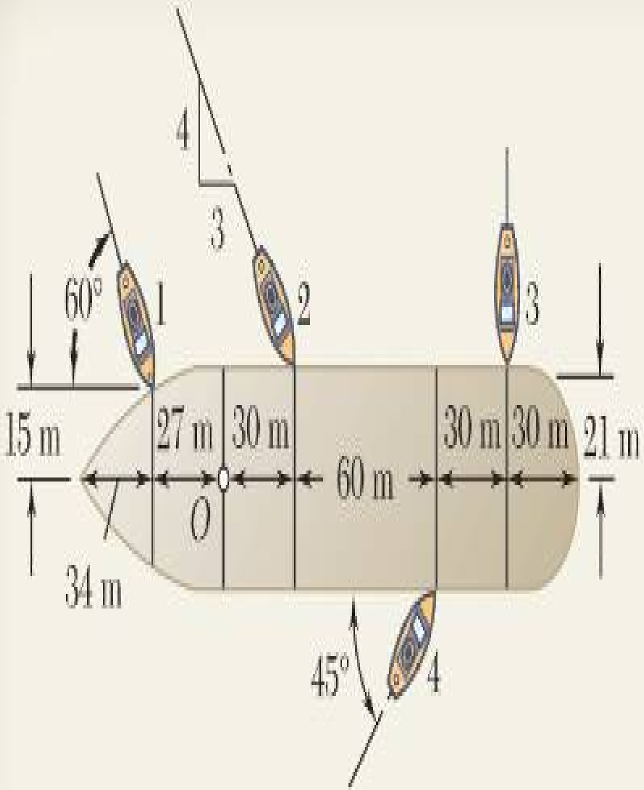
$$\underline{R} = \underline{F}_1 + \underline{F}_2 + \underline{F}_3 + \cdots = \sum \underline{F}$$

$$\underline{M}_O = \underline{M}_1 + \underline{M}_2 + \underline{M}_3 + \cdots = \sum (\underline{r} \times \underline{F})$$

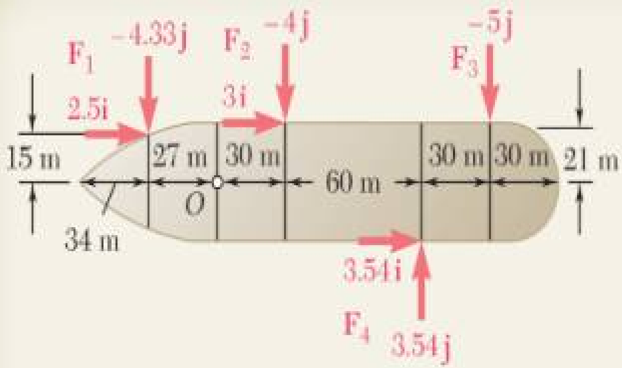
• The reduction of a system of coplanar forces to a force **R** at any point and a couple **M**, will be considerably simplified if the given forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$. etc., are resolved into their *x* and *y* components prior to moving them to the point.



SAMPLE PROBLEM 3.9



Four tugboats are used to bring an ocean liner to its pier. Each tugboat exerts a 5000-N force in the direction shown. Determine (*a*) the equivalent force-couple system at the foremast *O*, (*b*) the point on the hull where a single, more powerful tugboat should push to produce the same effect as the original four tugboats.

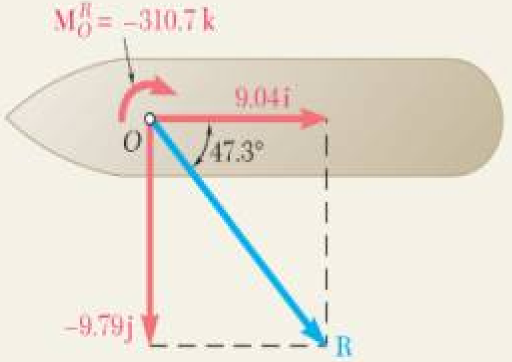


SOLUTION

a. Force-Couple System at O. Each of the given forces is resolved into components in the diagram shown (kN units are used). The force-couple system at O equivalent to the given system of forces consists of a force $\mathbf{R}$ and a couple $\mathbf{M}_O^R$ defined as follows:

$$\begin{aligned}\mathbf{R} &= \Sigma\mathbf{F}\\ &= (2.50\mathbf{i} - 4.33\mathbf{j}) + (3.00\mathbf{i} - 4.00\mathbf{j}) + (-5.00\mathbf{j}) + (3.54\mathbf{i} + 3.54\mathbf{j})\\ &= 9.04\mathbf{i} - 9.79\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{M}_O^R &= \Sigma(\mathbf{r} \times \mathbf{F})\\ &= (-27\mathbf{i} + 15\mathbf{j}) \times (2.50\mathbf{i} - 4.33\mathbf{j})\\ &\quad + (30\mathbf{i} + 21\mathbf{j}) \times (3.00\mathbf{i} - 4.00\mathbf{j})\\ &\quad + (120\mathbf{i} + 21\mathbf{j}) \times (-5.00\mathbf{j})\\ &\quad + (90\mathbf{i} - 21\mathbf{j}) \times (3.54\mathbf{i} + 3.54\mathbf{j})\\ &= (116.9 - 37.5 - 120 - 63 - 600 + 318.6 + 74.3)\mathbf{k}\\ &= -310.7\mathbf{k}\end{aligned}$$



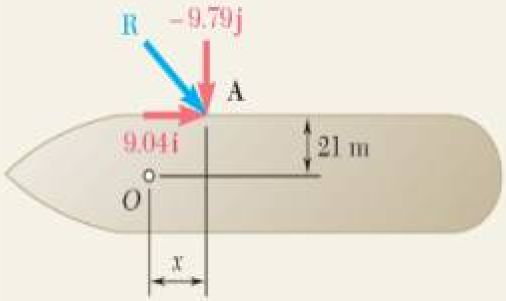
The equivalent force-couple system at O is thus

$$\mathbf{R} = (9.04 \text{ kN})\mathbf{i} - (9.79 \text{ kN})\mathbf{j} \qquad \mathbf{M}_O^R = -(310.7 \text{ kN}\cdot\text{m})\mathbf{k}$$

or

$$\mathbf{R} = 13.33 \text{ kN} \measuredangle 47.3° \qquad \mathbf{M}_O^R = 310.7 \text{ kN}\cdot\text{m} \curvearrowright$$

Remark. Since all the forces are contained in the plane of the figure, we could have expected the sum of their moments to be perpendicular to that plane. Note that the moment of each force component could have been obtained directly from the diagram by first forming the product of its magnitude and perpendicular distance to O and then assigning to this product a positive or a negative sign depending upon the sense of the moment.



b. Single Tugboat. The force exerted by a single tugboat must be equal to **R**, and its point of application A must be such that the moment of **R** about O is equal to $\mathbf{M}_O^R$. Observing that the position vector of A is

$$\mathbf{r} = x\mathbf{i} + 21\mathbf{j}$$

we write

$$\mathbf{r} \times \mathbf{R} = \mathbf{M}_O^R$$
$$(x\mathbf{i} + 21\mathbf{j}) \times (9.04\mathbf{i} - 9.79\mathbf{j}) = -310.7\mathbf{k}$$
$$-x(9.79)\mathbf{k} - 189.8\mathbf{k} = -310.7\mathbf{k} \qquad x = 12.3 \text{ m}$$

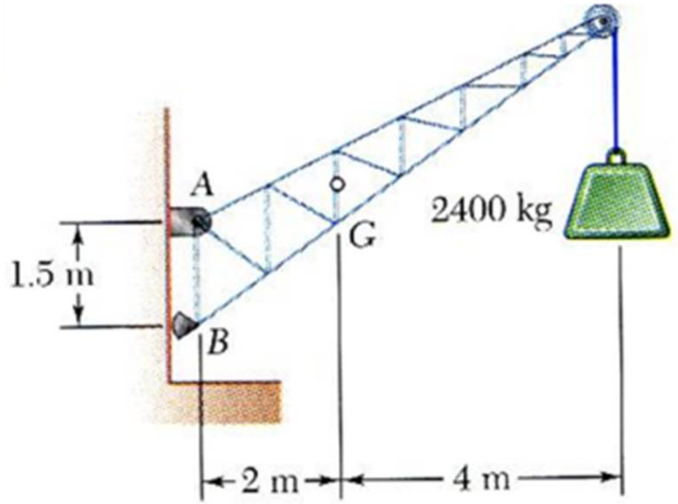
Equilibrium of Rigid Bodies

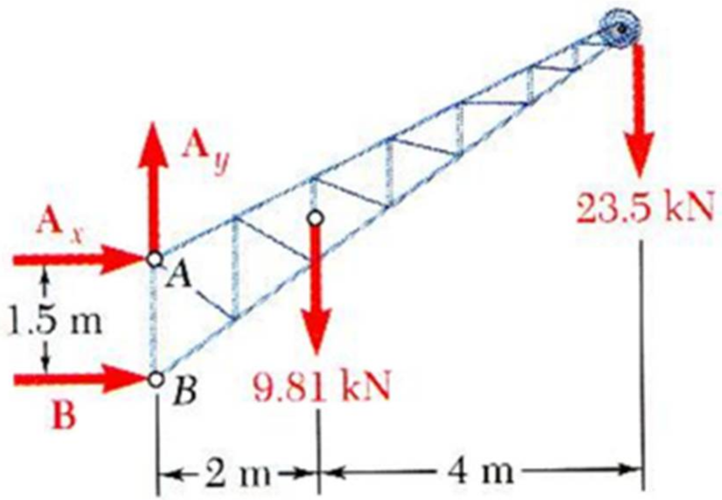
- A rigid body is said to be in equilibrium when the external forces acting on it form a system of forces equivalent to zero, i.e., a system which has no resultant force and no resultant couple.

- The necessary and sufficient conditions for equilibrium thus can be written as :

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_A = 0$$

Free-Body Diagram

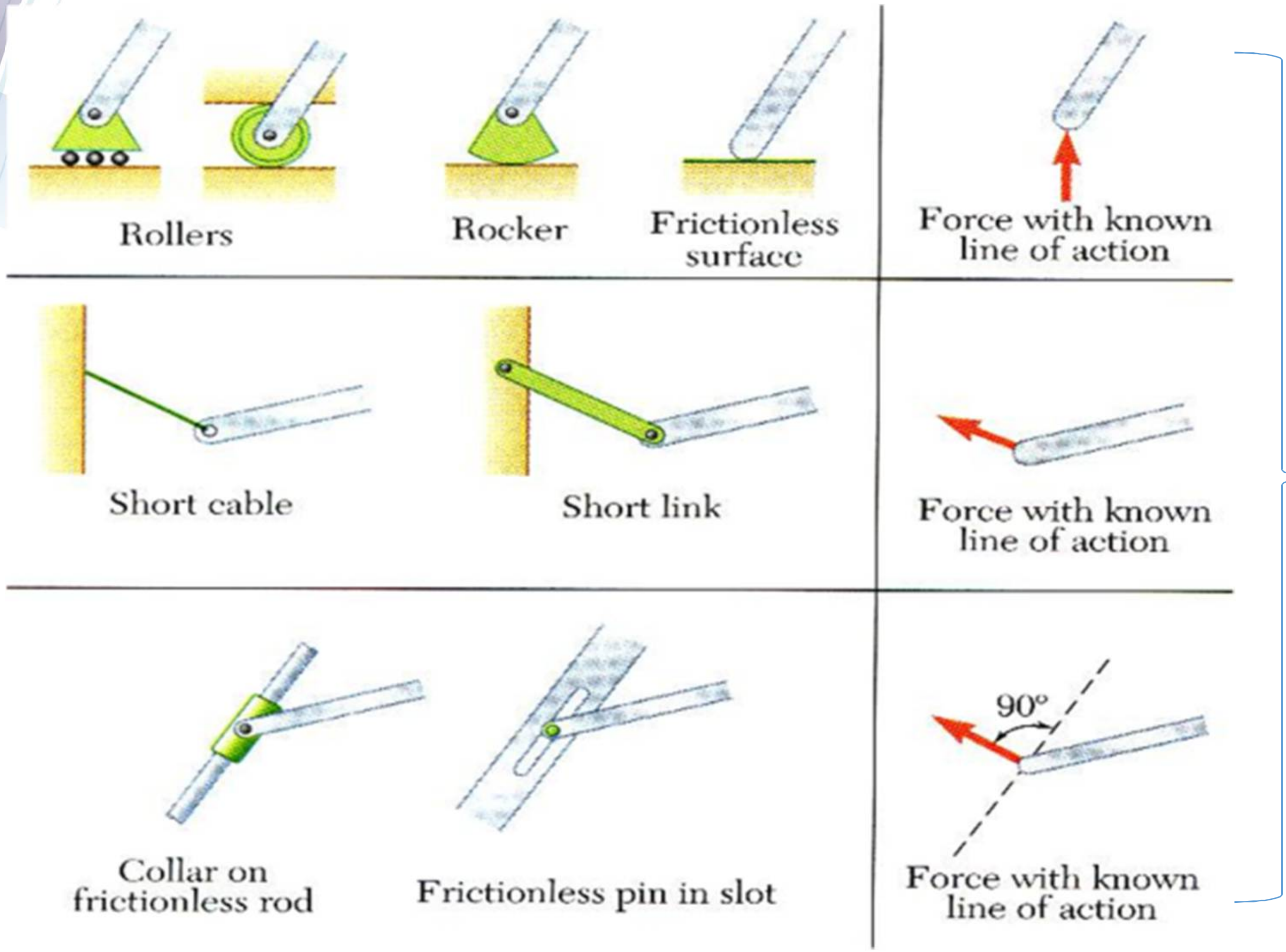




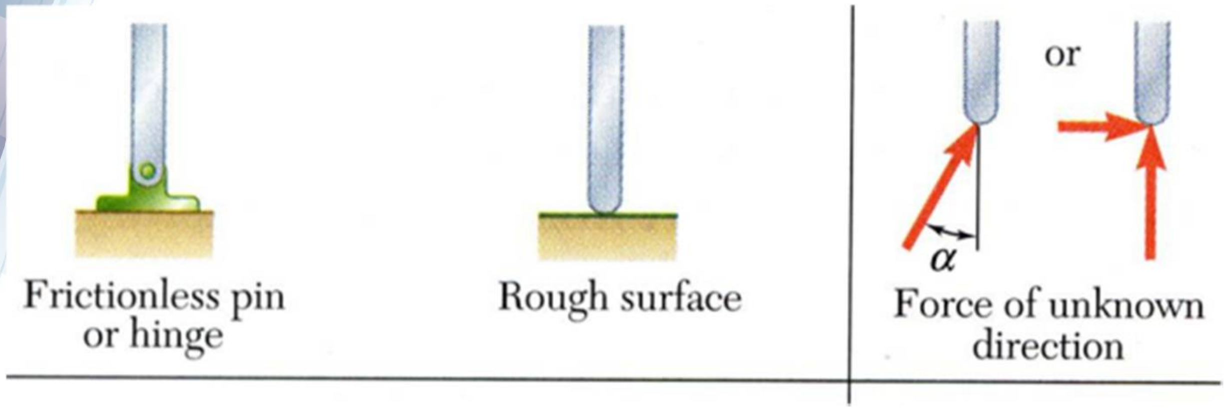
Procedure to draw free body diagram:

- Select the extent of the free-body and detach it from the ground and all other bodies.
- Indicate point of application, magnitude, and direction of external forces, including the rigid body weight.
- Indicate point of application and assumed direction of unknown applied forces. These usually consist of reactions through which the ground and other bodies oppose the possible motion of the rigid body.
- Include the dimensions necessary to compute the moments of the forces.

Common types of Reactions at Supports and Connections for a Two-Dimensional Structure



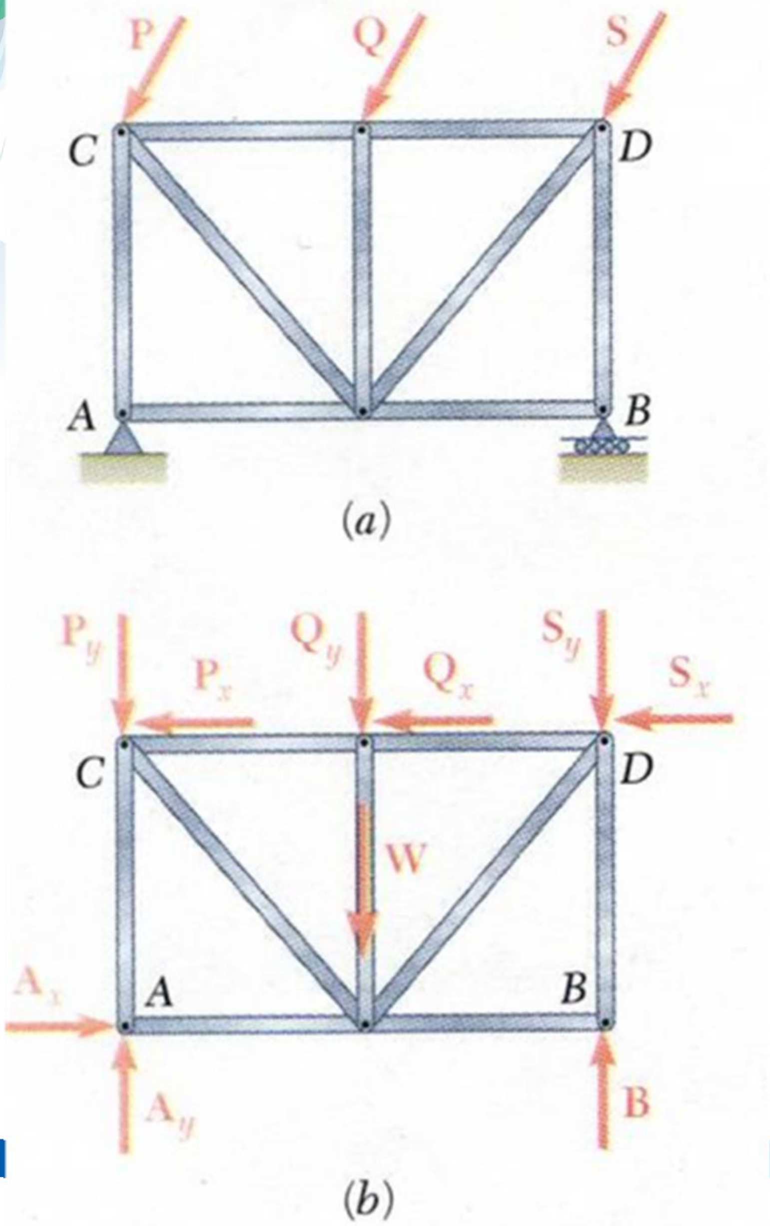
- Reactions equivalent to a force with known line of action.



- Reactions equivalent to a force of unknown direction and magnitude.

- Reactions equivalent to a force of unknown direction and magnitude and a couple.of unknown magnitude

Equilibrium of a Rigid Body in Two Dimensions



- Equations of equilibrium are

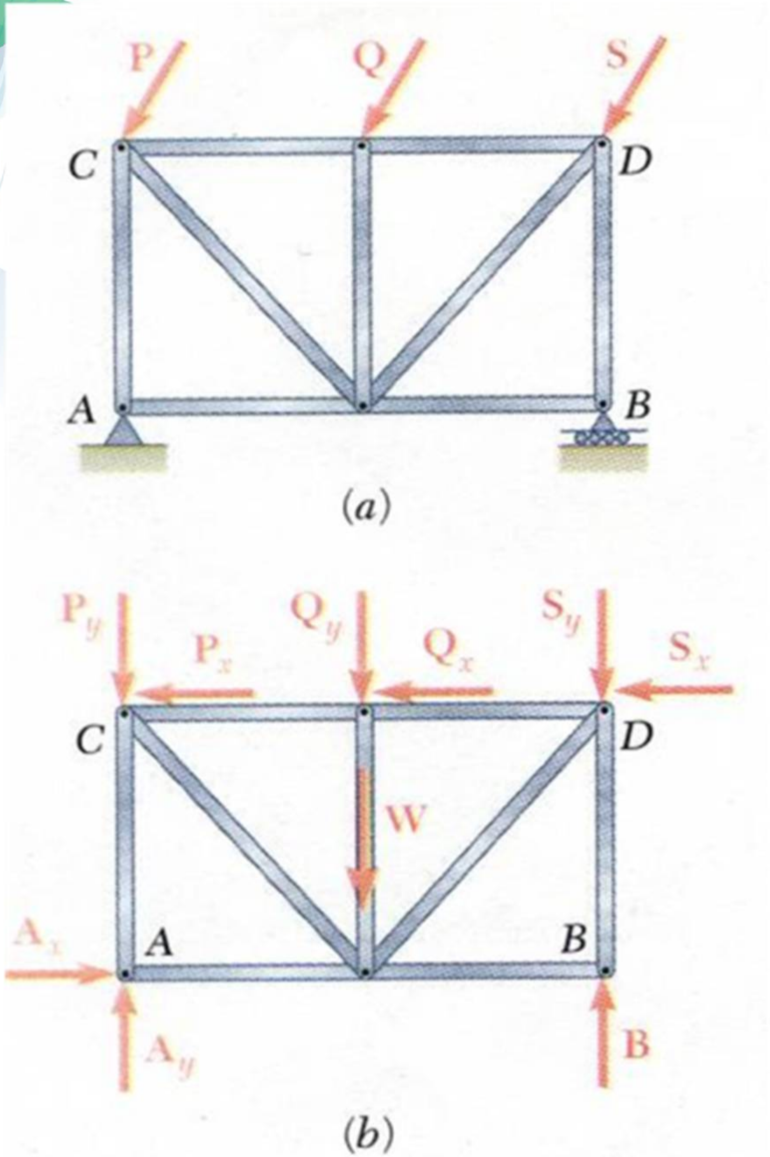
$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0$$

where *A* is any point in the plane of the structure.

- The 3 equations can be solved for no more than 3 unknowns.

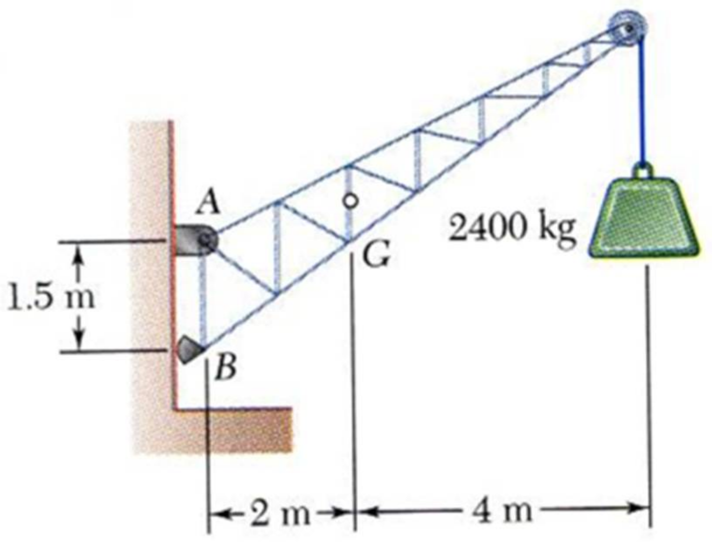
- The 3 equations can not be augmented with additional equations, but they can be replaced

$$\sum F_x = 0 \quad \sum M_A = 0 \quad \sum M_B = 0$$

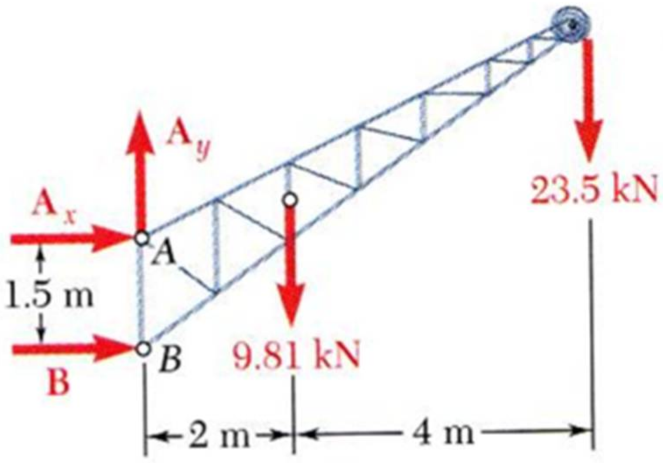


- Pin A = prevents point A frm moving by exerting Ax and Ay.
- Roller B=keeps the truss from rotating about A by exert By
- Sum $\sum M_A=0$
- Sum $\sum F_x=0$
- Sum $\sum F_y=0$

Sample Problem 4.1



A fixed crane has a mass of 1000 kg and is used to lift a 2400 kg crate. It is held in place by a pin at *A* and a rocker at *B*. The center of gravity of the crane is located at *G*. Determine the components of the reactions at *A* and *B*.

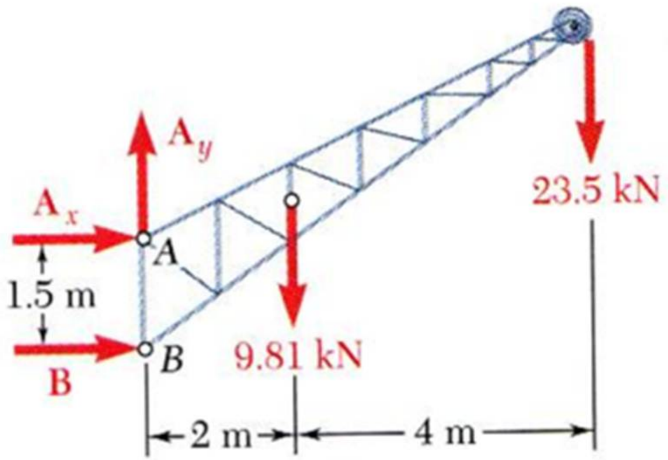


1. Draw the free-body diagram.

2. Determine B by solving the equation for the sum of the moments of all forces about A.

$$\sum M_A = 0: \quad +B(1.5\text{m}) - 9.81\,\text{kN}(2\text{m}) - 23.5\,\text{kN}(6\text{m}) = 0$$

$$\boxed{B = +107.1\,\text{kN}}$$



3. Determine the reactions at *A* by solving the equations for the sum of all horizontal forces and all vertical forces.

$$\sum F_x = 0: \quad A_x + B = 0$$

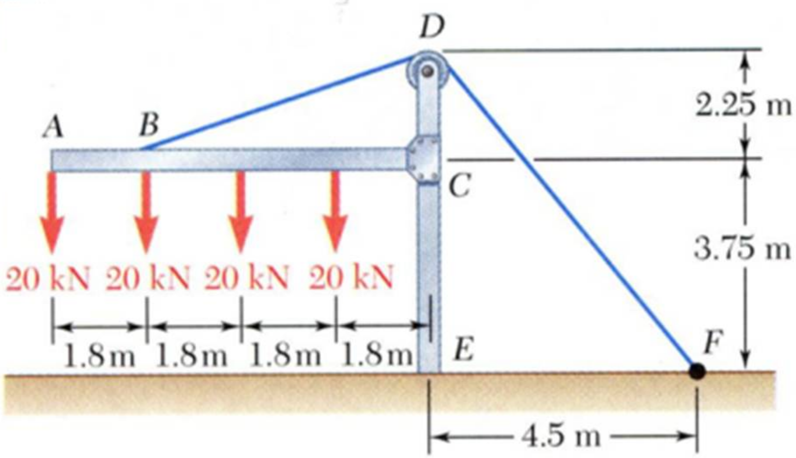
$$A_x = -107.1\,\text{kN}$$

$$\sum F_y = 0: \quad A_y - 9.81\,\text{kN} - 23.5\,\text{kN} = 0$$

$$A_y = +33.3\,\text{kN}$$

- Check the values obtained.

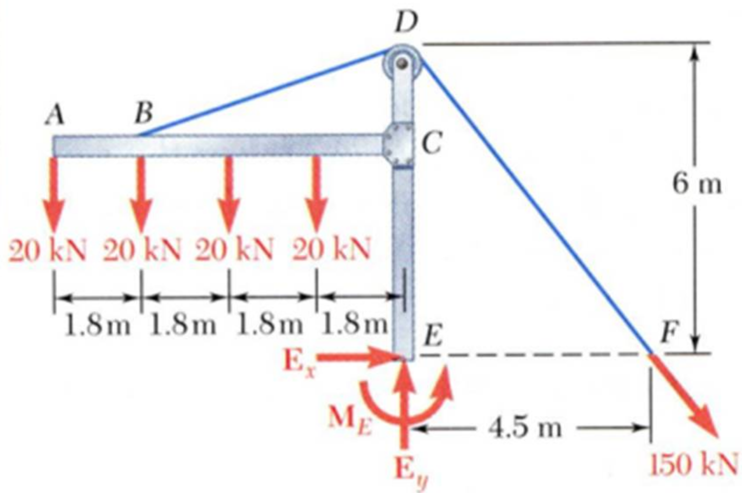
Sample Problem 4.4



The frame supports part of the roof of a small building. The tension in the cable is 150 kN.

Determine the reaction at the fixed end *E*.

Sample Problem 4.4



1. Draw the free-body diagram for the frame and cable.

2. Solve 3 equilibrium equations for the reaction force components and couple.

$$\sum F_x = 0: \quad E_x + \frac{4.5}{7.5}(150\,\text{kN}) = 0$$

$$E_x = -90.0\,\text{kN}$$

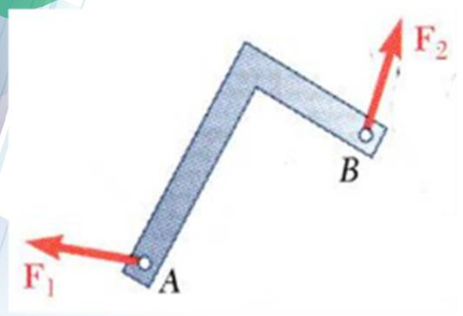
$$\sum F_y = 0: \quad E_y - 4(20\,\text{kN}) - \frac{6}{7.5}(150\,\text{kN}) = 0$$

$$E_y = +200\,\text{kN}$$

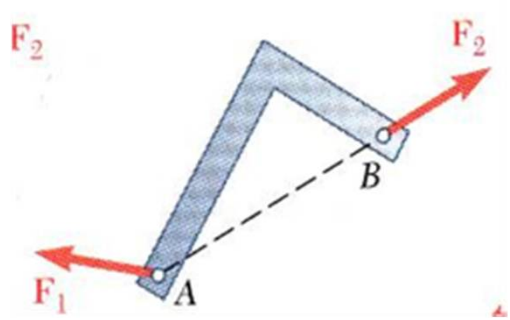
$$\sum M_E = 0: \quad +20\,\text{kN}(7.2\,\text{m}) + 20\,\text{kN}(5.4\,\text{m})$$
$$+20\,\text{kN}(3.6\,\text{m}) + 20\,\text{kN}(1.8\,\text{m})$$
$$-\frac{6}{7.5}(150\,\text{kN})4.5\,\text{m} + M_E = 0$$

$$M_E = 180.0\,\text{kN}\cdot\text{m}$$

Equilibrium of a Two-Force Body

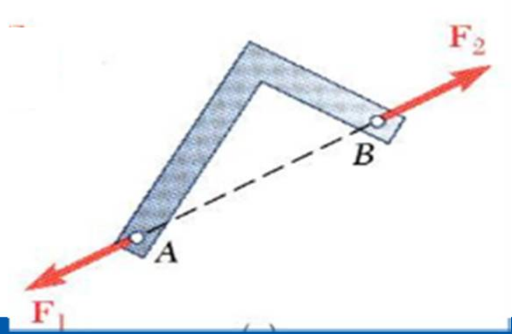


- Consider a plate subjected to two forces $\mathbf{F_1}$ and $\mathbf{F_2}$



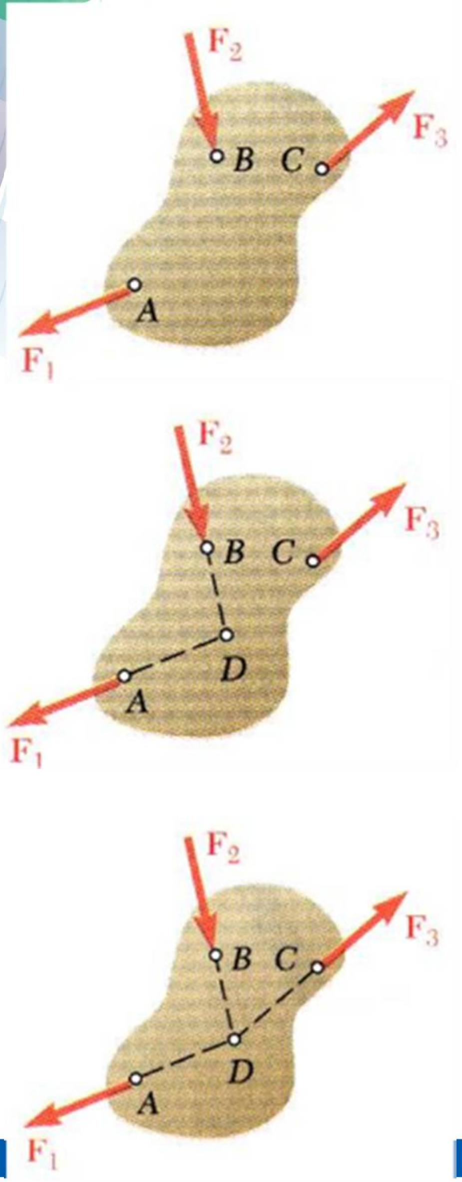
- For static equilibrium, the sum of moments about *A* must be zero. The moment of $\mathbf{F_2}$ must be zero. It follows that the line of action of $\mathbf{F_2}$ must pass through *A*.

- Similarly, the line of action of $\mathbf{F_1}$ must pass through *B* for the sum of moments about *B* to be zero.



- Requiring that the sum of forces in any direction be zero leads to the conclusion that $\mathbf{F_1}$ and $\mathbf{F_2}$ must have equal magnitude but opposite sense.

Equilibrium of a Three-Force Body



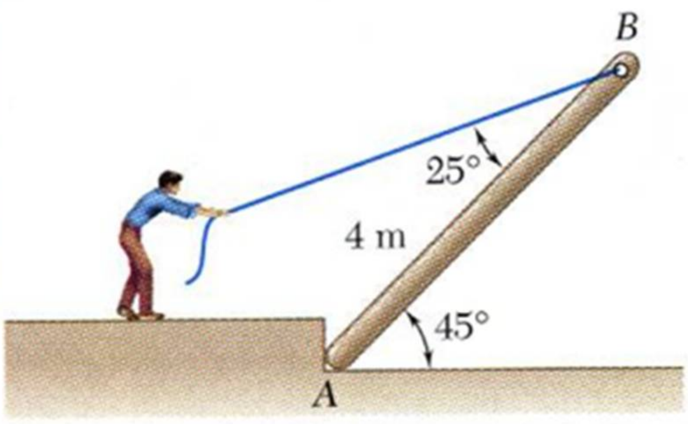
- Consider a rigid body subjected to forces acting at only 3 points.

- Assuming that their lines of action intersect, the moment of F_1 and F_2 about the point of intersection represented by D is zero.

- Since the rigid body is in equilibrium, the sum of the moments of F_1, F_2, and F_3 about any axis must be zero. It follows that the moment of F_3 about D must be zero as well and that the line of action of F_3 must pass through D.

- The lines of action of the three forces must be concurrent or parallel.

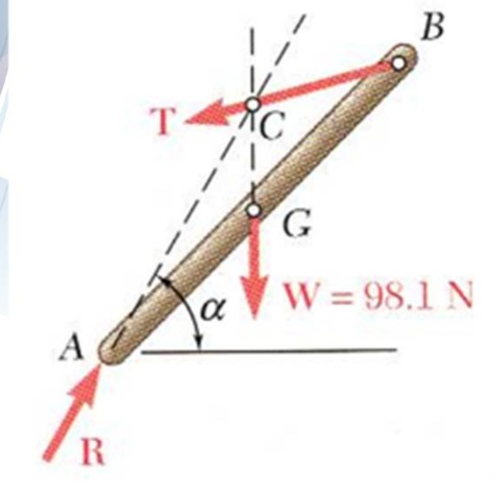
Sample Problem 4.6

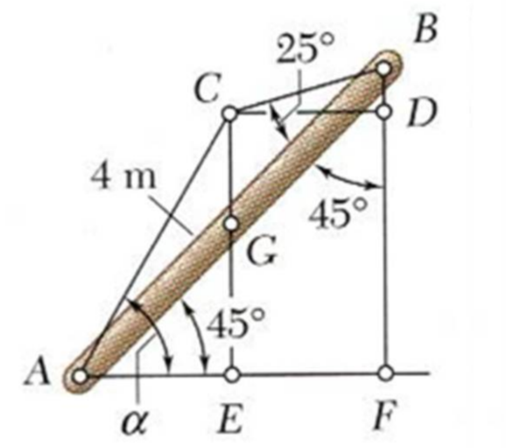


A man raises a 10 kg joist, of length 4 m, by pulling on a rope.

Find the tension in the rope and the reaction at *A*.

Sample Problem 4.6





1. Draw the free-body diagram of the joist.

2. Determine the direction of the reaction force **R**.

$$AF = AB\cos 45 = (4\,\text{m})\cos 45 = 2.828\,\text{m}$$

$$CD = AE = \tfrac{1}{2}AF = 1.414\,\text{m}$$

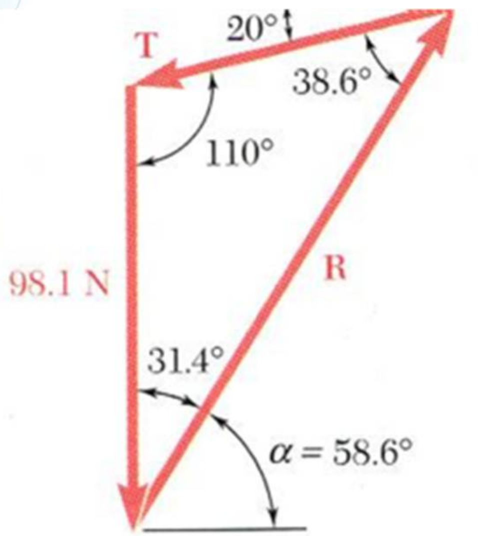
$$BD = CD\cot(45+20) = (1.414\,\text{m})\tan 20 = 0.515\,\text{m}$$

$$CE = BF - BD = (2.828 - 0.515)\,\text{m} = 2.313\,\text{m}$$

$$\tan\alpha = \frac{CE}{AE} = \frac{2.313}{1.414} = 1.636$$

$$\boxed{\alpha = 58.6^\circ}$$

Sample Problem 4.6



3. Determine the magnitude of the reaction force *R*.

$$\frac{T}{\sin 31.4^\circ} = \frac{R}{\sin 110^\circ} = \frac{98.1\,\text{N}}{\sin 38.6^\circ}$$

$$T = 81.9\,\text{N}$$

$$R = 147.8\,\text{N}$$

References:

1. Beer, Ferdinand P.; Johnston, E. Russell; “Vector Mechanics for Engineers - Statics”, 8th Ed., McGraw-Hill, Singapore, 2007.