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# BMA4723 VEHICLE DYNAMICS

## Ch4 Vehicle Equation of Motions

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# Chapter Description

- Aims
  - Explain the effect of the lateral force to the yaw moment and side-slip angle of the vehicle.
  - Analyse the effect of the steering angle to the vehicle motion.

- Expected Outcomes
  - Students are able to determine the effect of the lateral force to the yaw moment and side-slip angle of the vehicle.
  - Students are able to evaluate the effect of the steering angle to the vehicle motion.

- References

  - M.Abe, Vehicle Handling Dynamics Theory and Application, Second Edition, Published by Elsevier Ltd, 2015
  - Thomas D.Gillespie, Fundamental of Vehicle Dynamics, Published by Society of Automotive Engineers


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# Outlines

- 4.3 Effect of the Lateral Force to the Yaw Moment of the Vehicle
- 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

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# 4.3 Effect of the Lateral Force to the Yaw Moment of the Vehicle

- Fig.1 shows the lateral forces acting on the tires.
- Due to the lateral forces, the yaw moment is generated at the center of the vehicle.

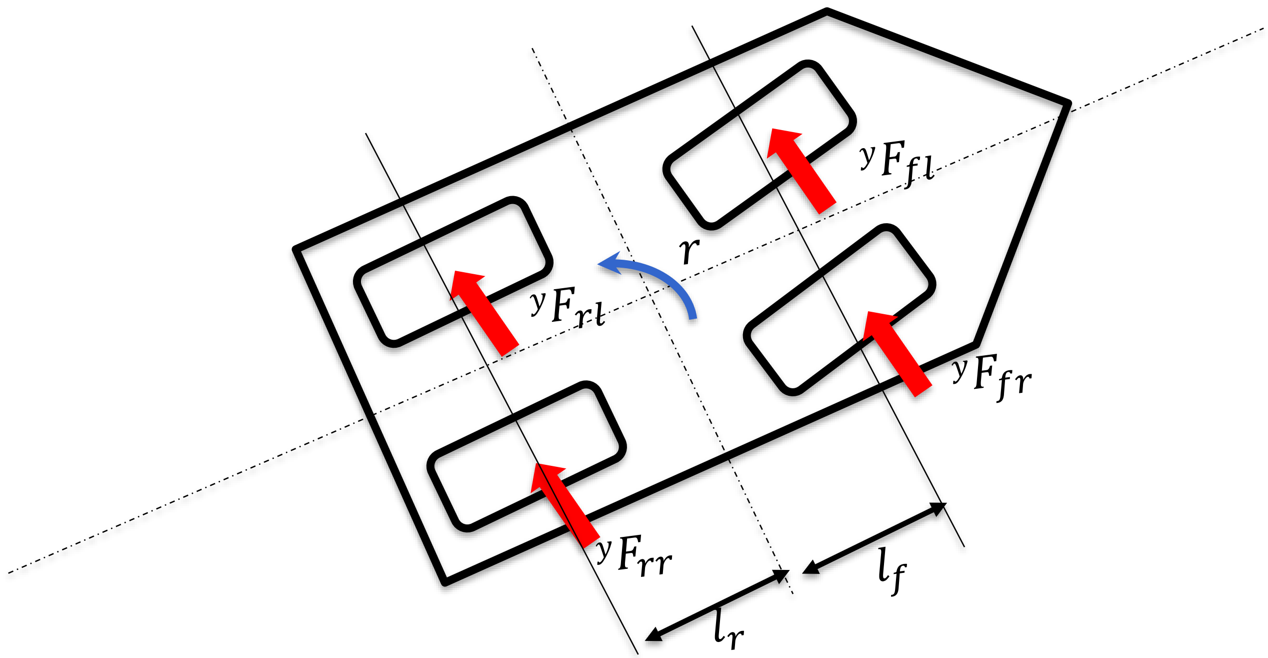


Figure 1 Lateral forces acting on the tires

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# 4.3 Effect of the Lateral Force to the Yaw Moment of the Vehicle

- Based on Fig.1, from the Newton's Second Law, the rotational motion of the vehicle can be expressed as:

$$I\frac{dr}{dt} = l_f({}^yF_{fr} + {}^yF_{fr}) - l_r({}^yF_{rr} + {}^yF_{rl}) \qquad \text{(Eq.1)}$$

where,
$I$: inertia of the vehicle
$r$: yaw moment at the center of the vehicle
$l_f$: length from the front axle to the center of the vehicle
$l_r$: length from the rear axle to the center of the vehicle
${}^yF$: lateral force at the front and rear tires

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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

- Lateral forces also generated the side-slip angle at each tire, $\beta_{fr}$, $\beta_{fl}$, $\beta_{rr}$ and $\beta_{rl}$.
- AS described in Chapter 2, the side-slip angle of the tire is the angle between tire heading direction and the tire travelling direction.

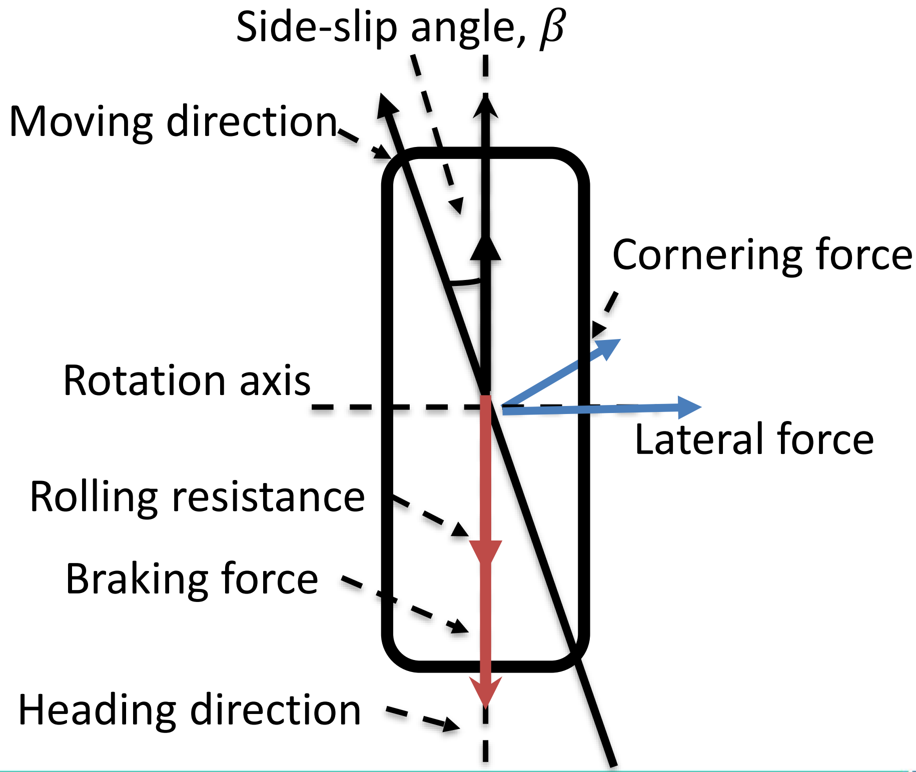


Figure 2 Side-slip angle at the tire

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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

The side-slip angle due to the lateral forces at each tire are illustrated in Fig.3

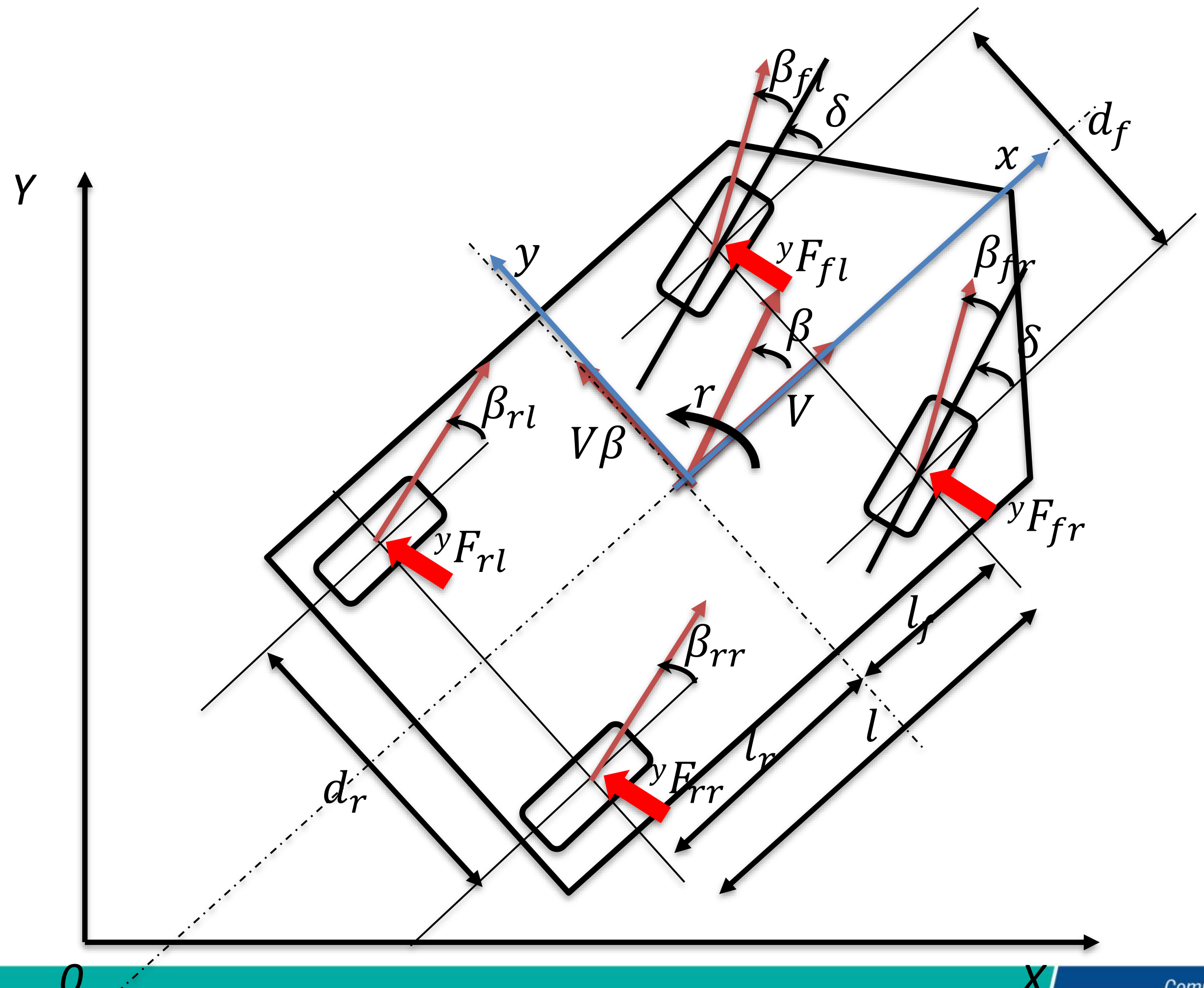


Figure 3 Side-slip angle at the tires

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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

- Based on Fig.3, the vehicle has the velocity component at the longitudinal axis, $x$ axis and lateral axis, $y$ axis.
- The velocity component at $x$ axis is $V$ and the velocity component at $y$ axis is $V\beta$.
- The yaw angular velocity at the center of gravity is $r$.
- During traction or braking, the motion of the tires also influenced by the velocity component of the vehicle ($V$ and $V\beta$) and yaw angular velocity of the vehicle ($r$).
- Fig. 4 shows the velocity component at each tire.

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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

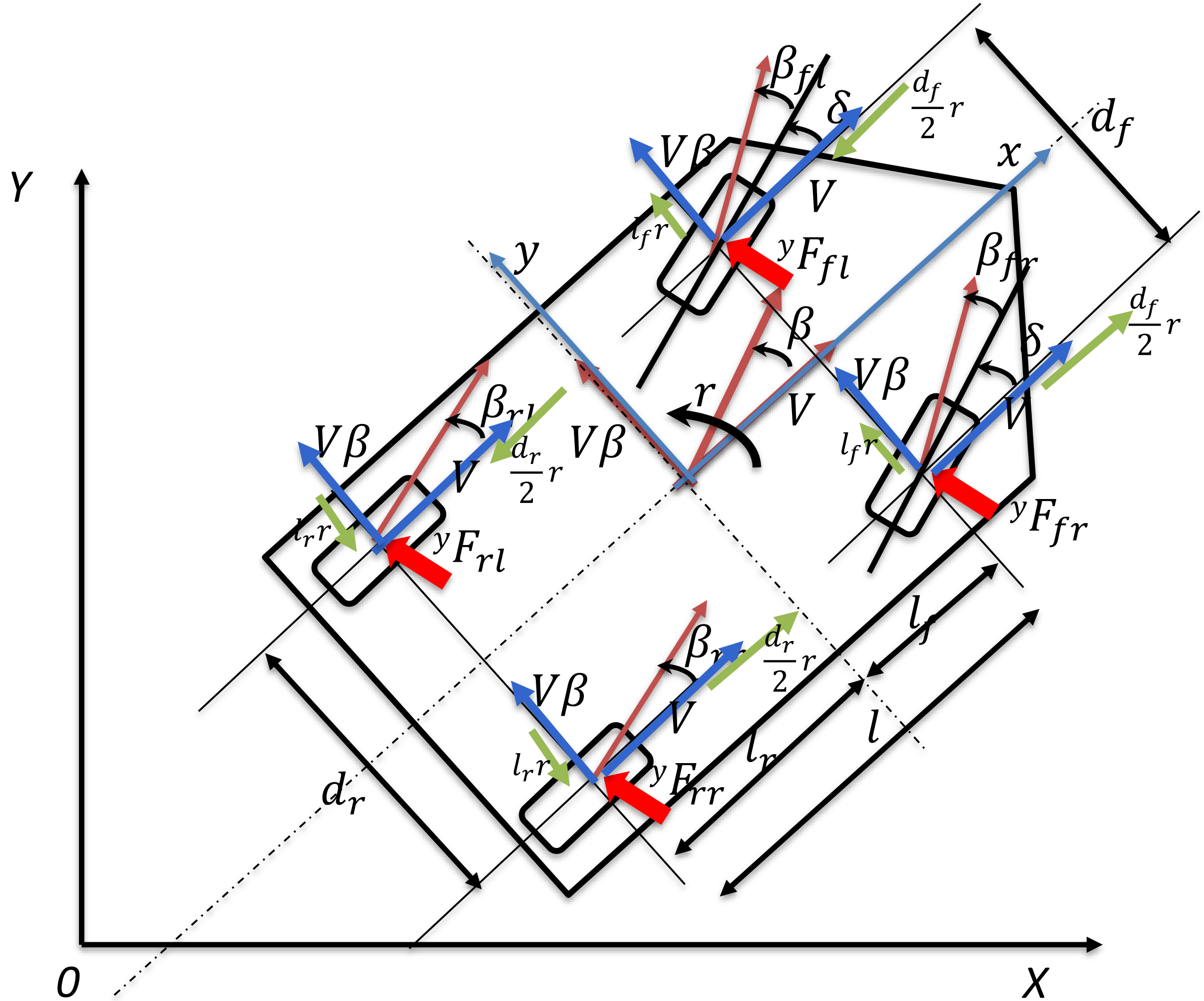

Figure 4 Velocity components at each tire

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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

- From Fig.4, the actual steer angle of the front tire with respect to the longitudinal axis, $x$ of the vehicle is $\delta$.
- This actual steer angle, $\delta$ is the heading direction of the tire.
- Then, due to the lateral forces and yaw moment of the vehicle, the front tires will have the side-slip angle.
- For the rear tires, the steering angle is 0 because no steering connection to the rear tires.
- Then, the heading direction of the tires is same as the longitudinal direction ($x$ axis) of the vehicle.
- Same as front tires, the rear tires also will have side-slip angle due to the lateral forces and yaw moment of the vehicle.

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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

- Based on Fig.4, the velocity component at each tire consists of velocity components of the vehicle ($V$ and $V\beta$), and yaw moment velocity of the vehicle ($r$).
- Fig.5 shows the velocity components at front left tires.

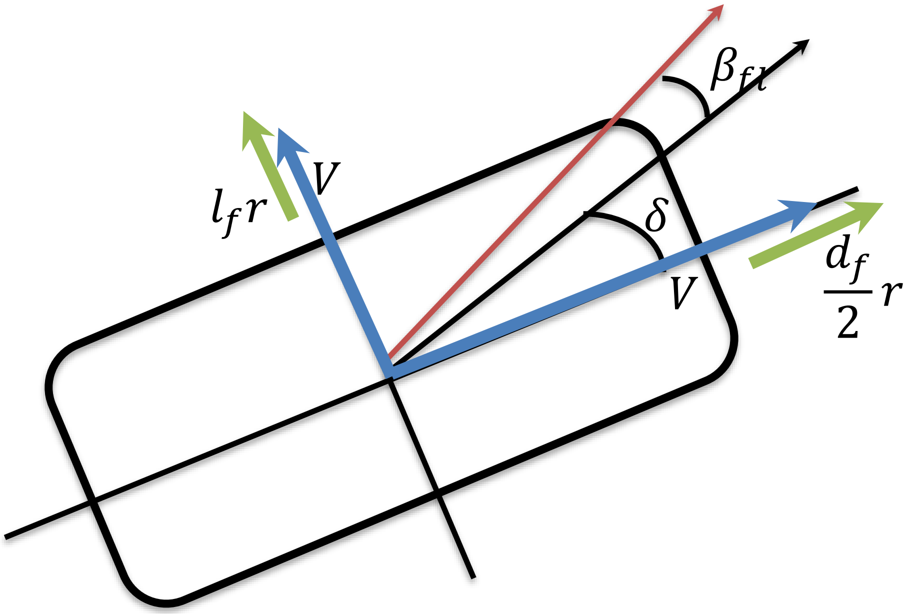


Figure 5 Velocity components at the front tire

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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

From the velocity components at the tires, the side-slip angle of each tire can be expressed as:

$$\beta_{fl} = \frac{V\beta + l_f r}{V - d_f r/2} - \delta \qquad \text{(Eq.2)}$$

$$\beta_{fr} = \frac{V\beta + l_f r}{V + d_f r/2} - \delta \qquad \text{(Eq.3)}$$

$$\beta_{rl} = \frac{V\beta - l_f r}{V - d_r r/2} \qquad \text{(Eq.4)}$$

$$\beta_{rr} = \frac{V\beta - l_f r}{V + d_r r/2} \qquad \text{(Eq.5)}$$

where $d_f$ and $d_r$ are the vehicle front and rear treads.

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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

From Eq.2 to Eq.4, $\left|l_f r/v\right|$, $|l_r r/v|$ , $\left|d_f r/2v\right|$, $|d_r r/2v| \ll 1$ can be ignored as negligible.

Then, the side-slip angle of the vehicle could be written as:

$$\beta_{fl} = \frac{V\beta + l_f r}{V - d_f r/2} - \delta \approx \beta + \frac{l_f}{V} r - \delta \qquad \text{(Eq.6)}$$

$$\beta_{fr} = \frac{V\beta + l_f r}{V + d_f r/2} - \delta \approx \beta + \frac{l_f}{V} r - \delta \qquad \text{(Eq.7)}$$

$$\beta_{rl} = \frac{V\beta - l_f r}{V - d_r r/2} \approx \beta - \frac{l_r}{V} r \qquad \text{(Eq.8)}$$

$$\beta_{rr} = \frac{V\beta - l_f r}{V + d_r r/2} \approx \beta - \frac{l_r}{V} \qquad \text{(Eq.9)}$$

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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

From Eq.2 to Eq.4, $|l_f r/v|$, $|l_r r/v|$ , $|d_f r/2v|$, $|d_r r/2v| \ll 1$ can be ignored as negligible.

Then, the side-slip angle of the vehicle could be written as:

$$\beta_{fl} = \frac{V\beta + l_f r}{V - d_f r/2} - \delta \approx \beta + \frac{l_f}{V} r - \delta \quad \text{(Eq.6)}$$

$$\beta_{fr} = \frac{V\beta + l_f r}{V + d_f r/2} - \delta \approx \beta + \frac{l_f}{V} r - \delta \quad \text{(Eq.7)}$$

$$\beta_{rl} = \frac{V\beta - l_f r}{V - d_r r/2} \approx \beta - \frac{l_r}{V} r \quad \text{(Eq.8)}$$

$$\beta_{rr} = \frac{V\beta - l_f r}{V + d_r r/2} \approx \beta - \frac{l_r}{V} \quad \text{(Eq.9)}$$

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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

- By using Eq.6 and Eq.7, the side-slip angle of the front left tire is same as the front right tire.

$$\beta_{fl} = \beta_{fr} = \beta + \frac{l_f}{V} r - \delta \qquad \text{(Eq.10)}$$

- Similar for the rear tires, the side-slip angle of the rear left tire is same as the rear right tire.

$$\beta_{rl} = \beta_{rr} = \beta - \frac{l_r}{V} \qquad \text{(Eq.11)}$$

As the side-slip angle of the left and right tire is same, the four-wheeled vehicle model could be transformed to the two-wheeled vehicle model, also known as bicycle model.

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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

- Fig.6 shows the two-wheeled vehicle model.

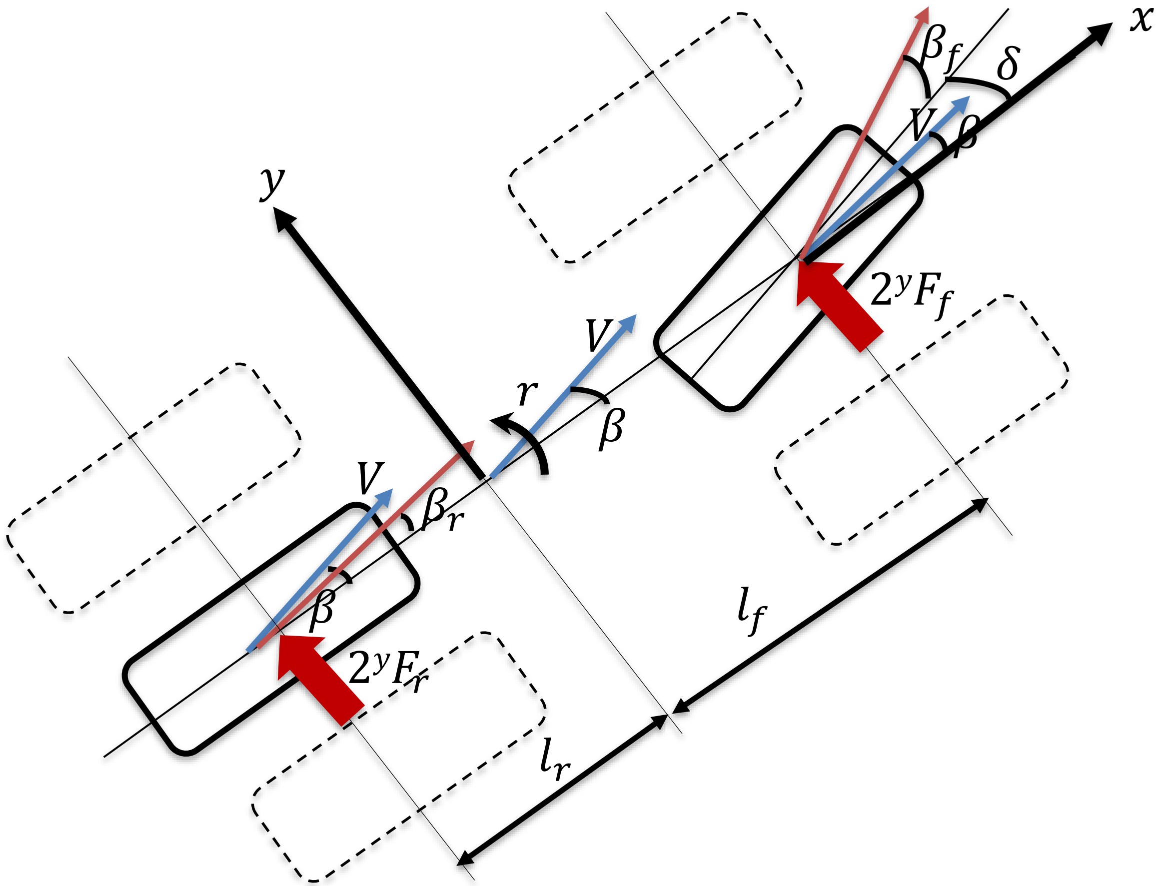


Figure 6 Bicycle model


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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

- From Fig.6, the lateral force at the front and rear tires are:

$$2{}^{y}F_f = {}^{y}F_{fl} + {}^{y}F_{fr} \qquad \text{(Eq.12)}$$

$$2{}^{y}F_r = {}^{y}F_{rl} + {}^{y}F_{rr} \qquad \text{(Eq.13)}$$

Then, by using the acceleration equation in Chapter 4.1 as below:

$$\ddot{R} = -V\beta(\dot{\beta} + r)i + V(\dot{\beta} + r)j$$

The equation of motion at the lateral axis could be written as:

$$mV\left(\frac{d\beta}{dt} + r\right) = 2{}^{y}F_f + 2{}^{y}F_r \qquad \text{(Eq.14)}$$

And yaw moment of the vehicle is:

$$I\frac{dr}{dt} = 2l_f{}^{y}F_f - 2l_r{}^{y}F_r \qquad \text{(Eq.15)}$$

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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

- The lateral forces at the front and rear tires ($2^yF_f$ and $2^yF_r$) are proportional to the product of the side-slip angle ($\beta_f$ and $\beta_r$) and cornering stiffness of the tires ($K_f$ and $K_r$) .
- From Fig.7, all angles are positive in the anticlockwise direction.
- That figure shows that if the side-slip angle $\beta$ is positive, the lateral forces ($2^yF_f$ and $2^yF_r$) , act in the negative $y$- direction.

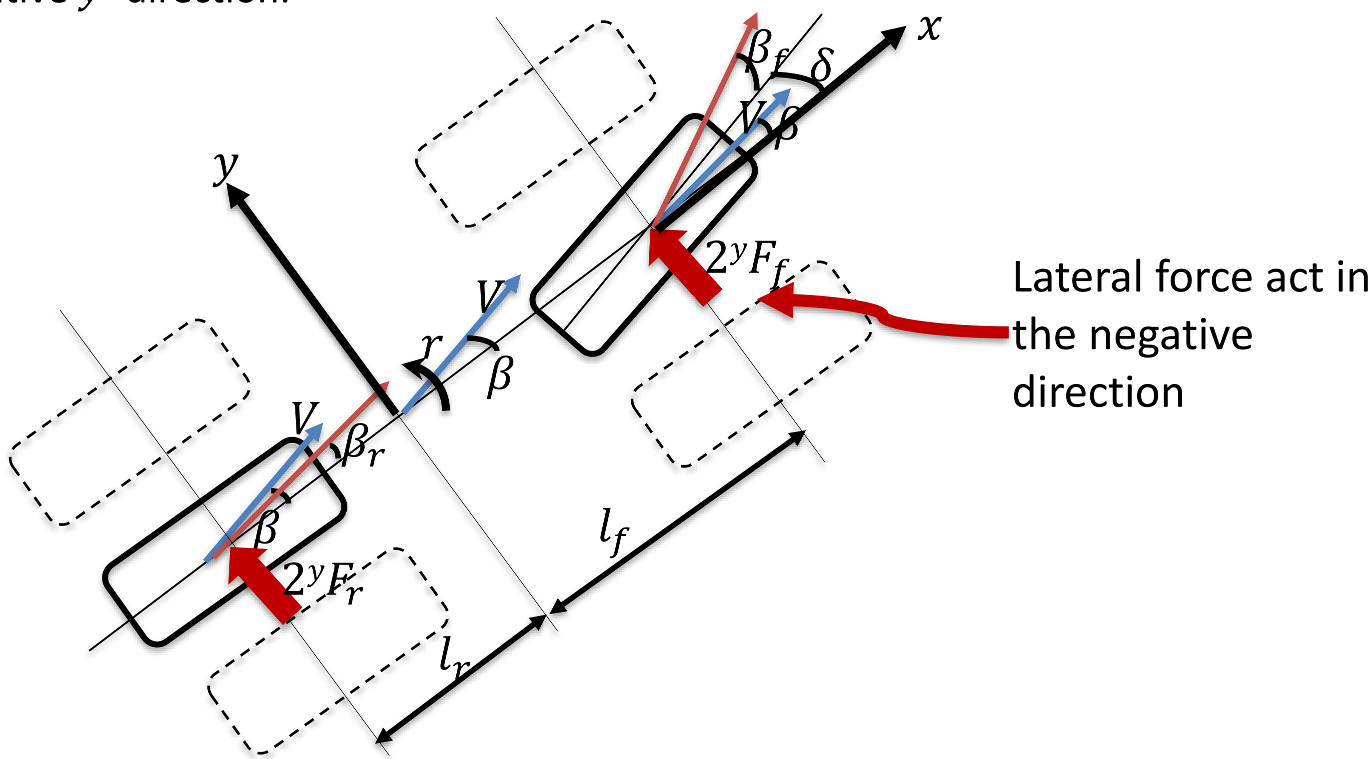


Figure 7 Lateral forces act in the negative y-direction

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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

- Then, the equation of the lateral forces are:

$$ {}^{y}F_f = -K_f\beta_f = -K_f(\beta + \frac{l_f}{V}r - \delta) \quad \text{(Eq.16)}$$

$$ {}^{y}F_r = -K_r\beta_r = -K_r(\beta - \frac{l_r}{V}) \quad \text{(Eq.17)}$$

Substitute Eq.16 and Eq.17 into Eq.14 and Eq.15 gives:

$$mV\left(\frac{d\beta}{dt} + r\right) = 2\,{}^{y}F_f + 2\,{}^{y}F_r$$

$$\boldsymbol{mV\left(\frac{d\beta}{dt} + r\right) = -2K_f(\beta + \frac{l_f}{V}r - \delta) - 2K_r(\beta - \frac{l_r}{V})} \quad \textbf{(Eq.18)}$$

$$I\frac{dr}{dt} = 2l_f\,{}^{y}F_f - 2l_r\,{}^{y}F_r$$

$$\boldsymbol{I\frac{dr}{dt} = -2l_fK_f(\beta + \frac{l_f}{V}r - \delta) + 2l_rK_r(\beta - \frac{l_r}{V})} \quad \textbf{(Eq.19)}$$

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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

- From Eq.18 and Eq.19, the steer angle is $\delta$.
- During driving, the input of the steer angle is come from the driver.
- The direction of the tire will follow the direction of the steering wheel.
- If the driver turn the steering wheel to the left side, the direction of the tire will change to the left side.
- On the other hand, if the driver turn the steering wheel to the right side, the tire will move to the right side.
- Fig.7 shows the direction of the steering wheel and the direction of the tire.

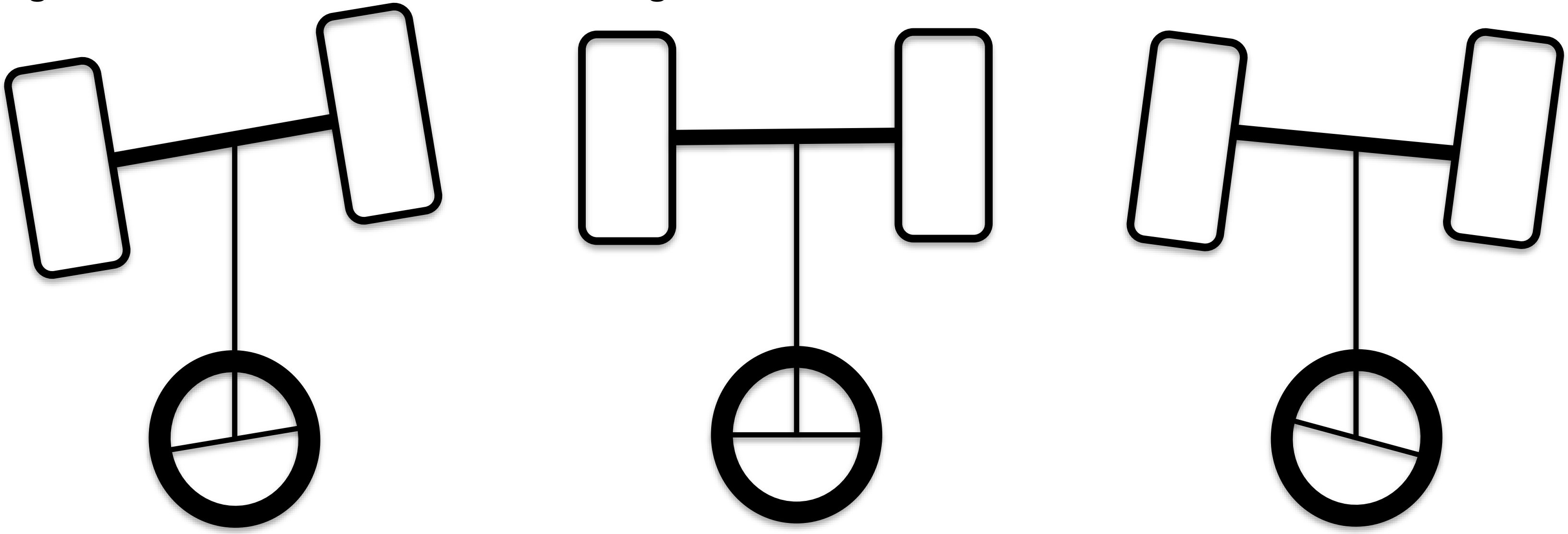

Figure 7 The direction of the tire correspond to the direction of the steering wheel

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# 4.4 Effect of the Lateral Force to the Side-slip Angle of the Tire

- Considered that the steering angle $\delta$ as an input, rearranging Eq.18 and Eq.19 gives:

$$mV\left(\frac{d\beta}{dt}\right) + 2(K_f + K_r)\beta + \left\{mV + \frac{2}{V}(l_f K_f - l_r K_r)\right\}r = 2K_f\delta$$

**(Eq.20)**

$$I\frac{dr}{dt} + 2(l_f K_f - l_r K_r)\beta + \frac{2({l_f}^2 K_f + {l_r}^2 K_r)}{V} r = 2l_f K_f \delta$$

**(Eq.21)**

The left-hand side of Eq.20 and Eq.21 shows the response of the vehicle (side-slip angle $\beta$ and yaw moment $r$) to the input, which is steering angle $\delta$.

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# Conclusion of The Chapter 4

- Conclusion #1
  - The yaw moment and side-slip angle are generated due to the lateral forces.

- Conclusion #2
  - The steering angle can effect the direction and motion of the vehicle.

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# Vehicle Dynamics

# Chapter 4

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