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Finite Element Analysis

Constant Strain Triangular Elements

by
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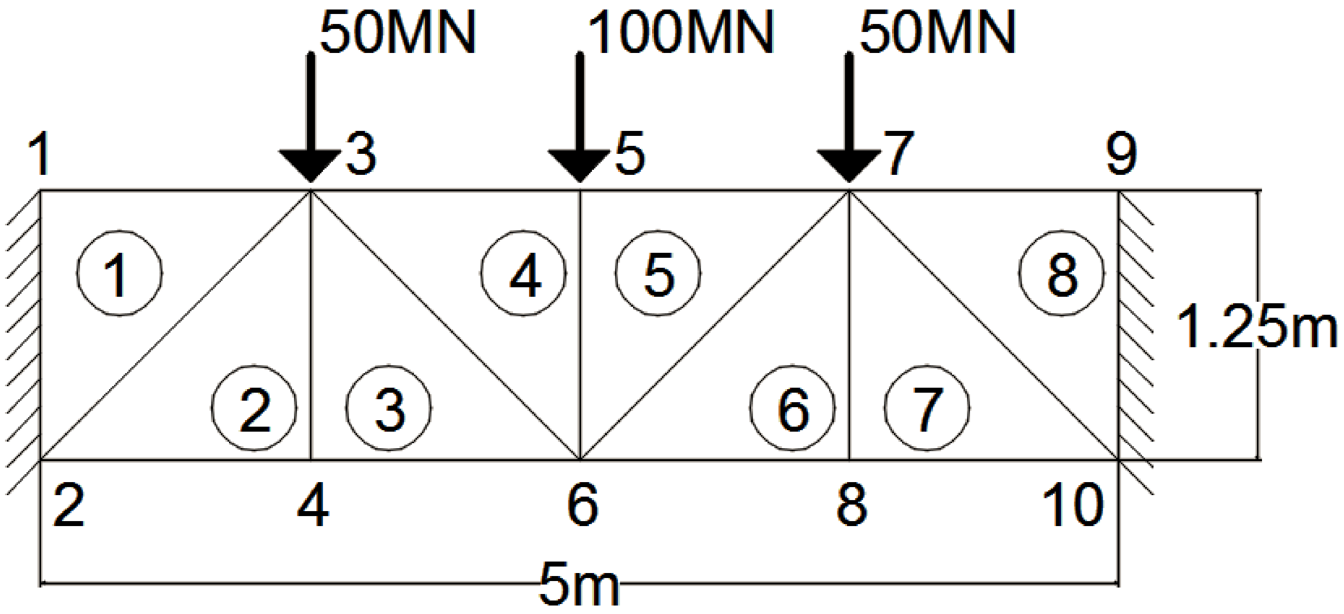
Lesson Outcomes

- At the end of this lesson, the student should be able to:
  - Apply the element equations of Constant Strain Triangle
  - Evaluate element stresses for a constant strain triangle element

Deep Beam Example

- A deep beam was discretized using constant strain triangular elements as shown. The modulus of elasticity for the beam material is 150GPa and the Poisson's ratio for the same is 0.2. The beam thickness is 250 mm. The nodal displacements for nodes 3 to 8 have been determined as given in **Table 1**.
- Generate the stiffness matrix for element No. 4.
- Evaluate the state of stresses in element No. 4 including direct stresses, shear stresses, principal stresses, and the orientation of the principal plane

Deep Beam



Element Deformations

Node Number	Displacement in x-direction	Displacement in y-direction
3	0.3408	-0.225
4	-0.2435	-0.2092
5	0.0029	-2.7462
6	-0.0038	-2.2161
7	-0.2424	-0.225
8	0.3058	-0.2092

Solution

Element No. 4 is connected to nodes 3, 6, and 5; in that order. Assuming origin to be at node 2:

$$x_1 = 1.25, \quad y_1 = 1.25$$

$$x_2 = 2.5, \quad y_2 = 0$$

$$x_3 = 2.5, \quad y_3 = 1.25$$

$$2A = 1.25(1.25 - 1.25) + 2.5(1.25 - 1.25) + 2.5(1.25 - 0) = 3.125\ m^2$$

$$\beta_1 = -1.25 \quad \beta_2 = 0 \quad \beta_3 = 1.25$$

$$\gamma_1 = 0 \quad \gamma_2 = -1.25 \quad \gamma_3 = 1.25$$

$$[B] = \begin{bmatrix} -0.4 & 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & -0.4 & 0 & 0.4 \\ 0 & -0.4 & -0.4 & 0 & 0.4 & 0.4 \end{bmatrix}$$

Solution (Continued)

$$E = 150\ GPa = 1.5 \times 10^{11}\ N/m^2$$

$$\nu = 0.2$$

$$\frac{E}{1-\nu^2} = 1.5625 \times 10^{11}$$

$$[D] = \begin{bmatrix} 1.5625 \times 10^{11} & 3.125 \times 10^{10} & 0 \\ 3.125 \times 10^{10} & 1.5625 \times 10^{11} & 0 \\ 0 & 0 & 6.25 \times 10^{10} \end{bmatrix}$$

$$[B]^T[D] = \begin{bmatrix} -0.4 & 0 & 0 \\ 0 & 0 & -0.4 \\ 0 & 0 & -0.4 \\ 0 & -0.4 & 0 \\ 0.4 & 0 & 0.4 \\ 0 & 0.4 & 0.4 \end{bmatrix} \begin{bmatrix} 1.5625 \times 10^{11} & 3.125 \times 10^{10} & 0 \\ 3.12 \times 10^{10} & 1.5625 \times 10^{11} & 0 \\ 0 & 0 & 6.25 \times 10^{10} \end{bmatrix}$$

Solution (Continued)

$$[B]^T[D] = \begin{bmatrix} -6.25\times10^{10} & -1.25\times10^{10} & 0 \\ 0 & 0 & -2.5\times10^{10} \\ 0 & 0 & -2.5\times10^{10} \\ -1.25\times10^{10} & -6.25\times10^{10} & 0 \\ 6.25\times10^{10} & 1.25\times10^{10} & 2.5\times10^{10} \\ 1.25\times10^{10} & 6.25\times10^{10} & 2.5\times10^{10} \end{bmatrix}$$

Solution (Continued)

$$[B]^T\,[D]\,[B] = \begin{bmatrix} -6.25\times10^{10} & -1.25\times10^{10} & 0 \\ 0 & 0 & -2.5\times10^{10} \\ 0 & 0 & -2.5\times10^{10} \\ -1.25\times10^{10} & -6.25\times10^{10} & 0 \\ 6.25\times10^{10} & 1.25\times10^{10} & 2.5\times10^{10} \\ 1.25\times10^{10} & 6.25\times10^{10} & 2.5\times10^{10} \end{bmatrix} \times$$

$$\begin{bmatrix} -0.4 & 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & -0.4 & 0 & 0.4 \\ 0 & -0.4 & -0.4 & 0 & 0.4 & 0.4 \end{bmatrix}$$

Solution (Continued)

$$[B]^T[D][B] = \begin{bmatrix} 2.5\times10^{10} & 0 & 0 & 5\times10^{9} & -2.5\times10^{10} & -5\times10^{9} \\ 0 & 1\times10^{10} & 1\times10^{10} & 0 & -1\times10^{10} & -1\times10^{10} \\ 0 & 1\times10^{10} & 1\times10^{10} & 0 & -1\times10^{10} & -1\times10^{10} \\ 5\times10^{9} & 0 & 0 & 2.5\times10^{10} & -5\times10^{9} & -2.5\times10^{10} \\ -2.5\times10^{10} & -1\times10^{10} & -1\times10^{10} & -5\times10^{9} & 3.5\times10^{10} & 1.5\times10^{10} \\ -5\times10^{9} & -1\times10^{10} & -1\times10^{10} & -2.5\times10^{10} & 1.5\times10^{10} & 3.5\times10^{10} \end{bmatrix}$$

$$[K] = tA[B]^T[D][B]$$

$$t = 250\ mm = 0.25\ m$$

$$A = 1.5625\ m^2$$

$$tA = 0.4\ m^3$$

Solution (Continued)

$$[K] = \begin{bmatrix} 1\times10^{10} & 0 & 0 & 2\times10^{9} & -1\times10^{10} & -2\times10^{9} \\ 0 & 4\times10^{9} & 4\times10^{9} & 0 & -4\times10^{9} & -4\times10^{9} \\ 0 & 4\times10^{9} & 4\times10^{9} & 0 & -4\times10^{9} & -4\times10^{9} \\ 2\times10^{9} & 0 & 0 & 1\times10^{10} & -2\times10^{9} & -1\times10^{10} \\ -1\times10^{10} & -4\times10^{9} & -4\times10^{9} & -2\times10^{9} & 1.4\times10^{10} & 6\times10^{9} \\ -2\times10^{9} & -4\times10^{9} & -4\times10^{9} & -1\times10^{10} & 6\times10^{9} & 1.4\times10^{10} \end{bmatrix}$$

$$[D]\,[B] = \begin{bmatrix} 1.5625\times10^{11} & 3.125\times10^{10} & 0 \\ 3.125\times10^{10} & 1.5625\times10^{11} & 0 \\ 0 & 0 & 6.25\times10^{10} \end{bmatrix} \times$$

$$\begin{bmatrix} -0.4 & 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & -0.4 & 0 & 0.4 \\ 0 & -0.4 & -0.4 & 0 & 0.4 & 0.4 \end{bmatrix}$$

Solution (Continued)

$$[D]\,[B] = \begin{bmatrix} -6.25\times10^{10} & 0 & 0 & -1.25\times10^{10} & 6.25\times10^{10} & 1.25\times10^{10} \\ -1.25\times10^{10} & 0 & 0 & -6.25\times10^{10} & 1.25\times10^{10} & 6.25\times10^{10} \\ 0 & -2.5\times10^{10} & -2.5\times10^{10} & 0 & 2.5\times10^{10} & 2.5\times10^{10} \end{bmatrix}$$

From Table 1, the deformation vector for element No. 4 is determined as:

$$\{d\} = \begin{Bmatrix} 3.408\times10^{-4} \\ -2.25\times10^{-4} \\ -3.8\times10^{-6} \\ -2.2161\times10^{-3} \\ 2.9\times10^{-6} \\ -2.7462\times10^{-3} \end{Bmatrix}$$

$$\{\sigma\} = [D]\,[B]\,\{d\}$$

After multiplying, we get:

Solution (Continued)

$$\{\sigma\} = \begin{Bmatrix} -27745000 \\ -37355000 \\ -62862500 \end{Bmatrix}$$

$$\sigma_x = -27745000\ N/m^2$$

$$\sigma_y = -37355000\ N/m^2$$

$$\tau_{xy} = -62862500\ N/m^2$$

Principal stresses:

$$\sigma_1 = \frac{-27745000 - 37355000}{2} + \sqrt{\left(\frac{-27745000 + 37355000}{2}\right)^2 + (-62862500)^2}$$

$$\Rightarrow \sigma_1 = 30495872\ N/m^2$$

$$\sigma_2 = \frac{-27745000 - 37355000}{2} - \sqrt{\left(\frac{-27745000 + 37355000}{2}\right)^2 + (-62862500)^2}$$

$$\Rightarrow \sigma_2 = -95595872\ N/m^2$$

Solution (Continued)

Orientation of the principal plane:

$$\tan 2\theta_p = \frac{-2 \times 62862500}{-27745000 + 37355000}$$

$$\Rightarrow \theta_p = -42.81^o$$

Author Information

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