Universiti Malaysia PAHANG
Engineering • Technology • Creativity

Finite Element Analysis

Formulation of a Beam Element

by
Dr. Gul Ahmed Jokhio
Faculty of Civil Engineering and Earth Resources
jokhio@ump.edu.my

Lesson Outcomes

- At the end of this lesson, the student should be able to:
  - Understand the discretization of a beam element
  - Develop stiffness equations for beam element

Beam Element

- A beam element has 2 degrees of freedom (DOF) per node
- Each node can either rotate (due to bending in the beam) or move in the transverse direction to the main axis of the beam (due to shear deformation in the beam
- The rotational degree of freedom is denoted by ϕ and the corresponding force (moment in this case) is denoted by m
- The translational degree of freedom is denoted by v and the corresponding force is denoted by f
- The x-axis is assumed to be aligned with the main axis of the beam, therefore, the rotational DOF is rotation about z-axis and translational DOF is translation in y-direction
- The sign convention used for beam elements is give next

Sign Convention for a Beam Element

- Moments are positive counterclockwise
- Rotations are positive counterclockwise
- Forces are positive in positive y direction
- Displacements are positive in positive y direction

Displacement Function

- A displacement function of any order can be assumed for a beam element, however, a function of order 1 is not appropriate
- A quadratic function might work but it will give only rough approximations as we know that the deflected curves of beams are not exactly circular
- Increasing the order of the displacement function will continue to improve the accuracy of results
- Higher order displacement functions will, however, increase the computational cost as well
- As a starting point, a cubic displacement function is assumed here

Displacement Function (Continued)

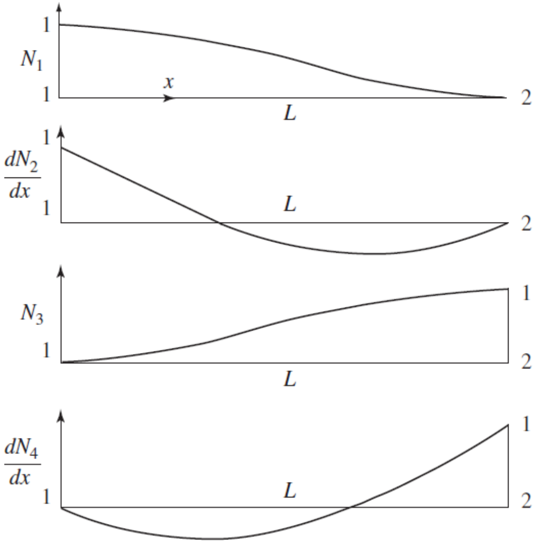
- Assuming the cubic displacement function:
- $v(x) = a_1x^3 + a_2x^2 + a_3x + a_4$
- The derivative of this function (slope) is given as:
- $\frac{dv}{dx} = 3a_1x^2 + 2a_2x + a_3$
- Inserting the nodal values for nodes 1 and 2, we get:
- $v(0) = v_1 = a_4$
- $\frac{dv(0)}{dx} = \phi_1 = a_3$
- $v(L) = v_2 = a_1L^3 + a_2L^2 + a_3L + a_4$
- $\frac{dv(L)}{dx} = \phi_2 = 3a_1L^2 + 2a_2L + a_3$

Displacement Function (Continued)

- Substituting the coefficients in the original function:
- $v = \left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^3 + \left[-\frac{3}{L^2}(v_1 -\right.$

Shape Functions

- From above, the shape functions can be obtained as:
- $N_1 = \frac{1}{L^3}(2x^3 - 3x^2L + L^3)$
- $N_2 = \frac{1}{L^3}(x^3L - 2x^2L^2 + xL^3)$
- $N_3 = \frac{1}{L^3}(-2x^3 + 3x^2L)$
- $N_4 = \frac{1}{L^3}(x^3L - x^2L^2)$



Strain-Displacement and Stress-Strain Relationships

- The strain of a beam element can be expressed as:
- $\varepsilon_x(x, y) = \frac{du}{dx}$
- Where u is the axial displacement function
- $u = -y\frac{dv}{dx}$
- $\varepsilon_x(x, y) = -y\frac{d^2v}{dx^2}$
- We know that
- $\frac{d^2v}{dx^2} = \frac{M}{EI}$, and $\sigma_x = E\varepsilon_x$

Strain-Displacement Relationships (Continued)

- $\sigma_x = \frac{-My}{I}$
- From elementary beam theory:
- $m(x) = EI\frac{d^2v}{dx^2}$
- $V = EI\frac{d^3v}{dx^3}$

Element Equations

- Element equations for the 4 force components, therefore, can be obtained as:
- $f_{1y} = V = EI\frac{d^3v(0)}{dx^3} = \frac{EI}{L^3}(12v_1 + 6L\phi_1 - 12v_2 + 6L\phi_2)$
- $m_1 = -m = -EI\frac{d^2v(0)}{dx^2} = \frac{EI}{L^3}(6Lv_1 + 4L^2\phi_1 - 6Lv_2 + 2L^2\phi_2)$
- $f_{2y} = -V = -EI\frac{d^3v(L)}{dx^3} = \frac{EI}{L^3}(-12v_1 - 6L\phi_1 + 12v_2 - 6L\phi_2)$
- $m_2 = m = EI\frac{d^2v(L)}{dx^2} = \frac{EI}{L^3}(6Lv_1 + 2L^2\phi_1 - 6Lv_2 + 4L^2\phi_2)$
- These can be expressed in matrix form as given next

Element Equations (Continued)

- Beam element equations in the matrix form are given as:

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Stiffness Matrix

- From the element equations, the stiffness matrix for a beam element is obtained as:

$$\frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}$$

Author Information

Dr. Gul Ahmed Jokhio

is a Senior Lecturer at FKASA, UMP. He completed his PhD from Imperial College London in 2012.