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Finite Element Analysis

Formulation of a Spring Element

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Lesson Outcomes

- At the end of this lecture, the student should be able to:
  - Understand the 8 steps of Finite Element Analysis
  - Apply the 8 steps of Finite Element Analysis to 2DOF structural systems
  - Formulate stiffness matrix for a spring element
  - Use the spring element formulation to analyze structures consisting of spring type members
  - Extract and interpret the results obtained from finite element analysis for spring elements

Example Structure (Spring)

- Consider the structure shown in Figure 1
- It consists of two springs connected end-to-end with a 6kN force pulling each spring
- The left end of the structure is 'fixed'
- The stiffness of each of the springs is given as '21 kN/m'
- Now, we will use the 8 steps of FEA to analyze this structure

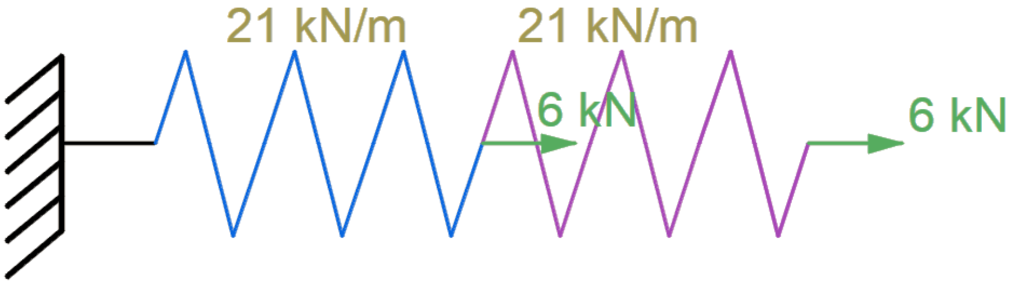


Figure 1

Step 1(a): Discretization

- This structure can be conveniently discretized into 3 nodes and 2 elements with element 1 connected to nodes 1 and 2 and element 2 connected to nodes 2 and 3
- It is conventional to identify nodes with numbers written close to the nodes
- The elements are also identified with numbers; to distinguish element numbers from node numbers, these are generally enclosed in circles
- The discretized structure is shown in Figure 2

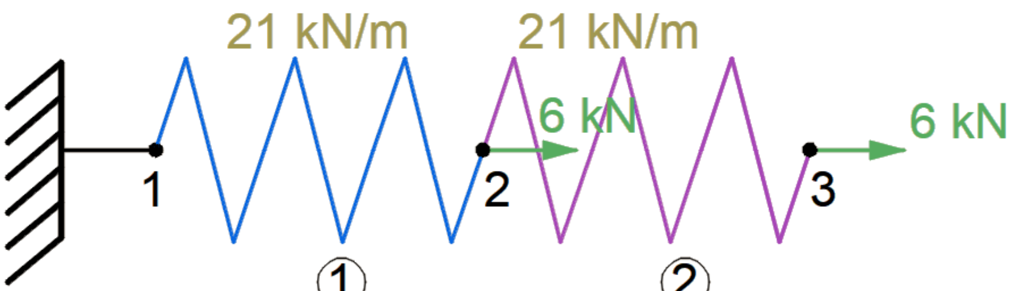


Figure 2

Step 1(b): Element Type

- To model this structure, it is best to use one dimensional spring elements with 2 nodes as shown in Figure 3
- The element type, therefore, is ‘linear spring’
- The following assumptions are made for this element:
- The spring obeys Hooke’s Law
- Resists forces only in the direction of the spring

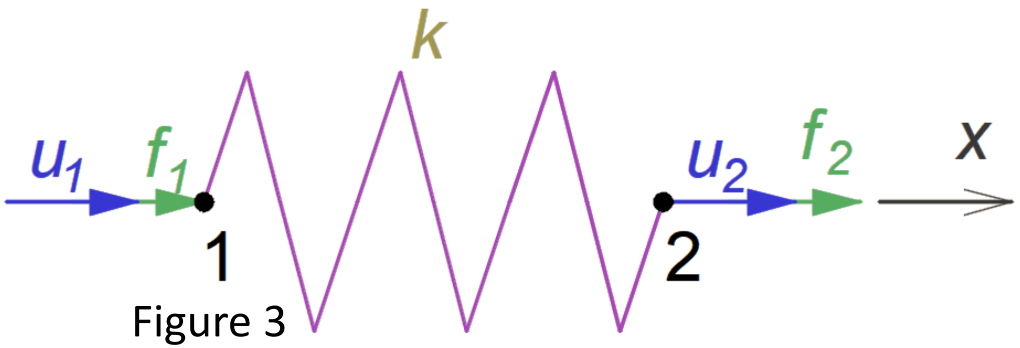


Figure 3

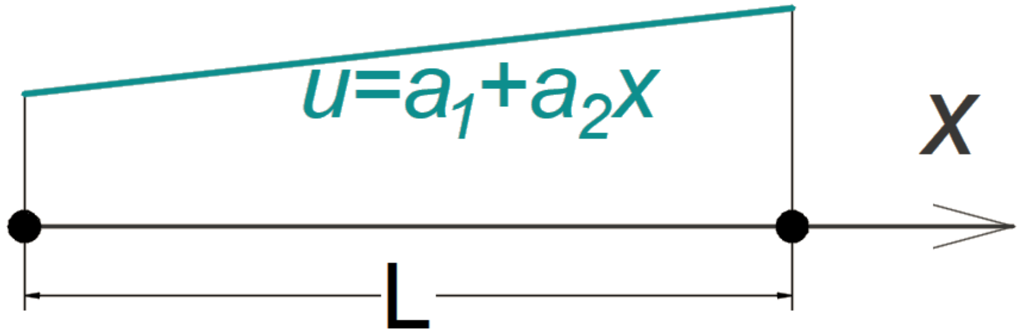
Step 1(b): Element Type (Continued)

- Points 1 and 2 are the ‘nodes’
- f_1 and f_2 are the forces in local x-direction
- u_1 and u_2 are the local nodal displacements (also called nodal *degrees of freedom*)
- k is the spring constant or *stiffness* of the spring
- x is taken positive going from node 1 to node 2

Step 2: Displacement Function

- A linear displacement function can be assumed for this element, shown in Figure 4 and given below:
- $u = a_1 + a_2 x$
- It can be expressed in matrix form as:
- $u = \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$

Figure 4



Step 2: Displacement Function (Continued)

- We can input the nodal values to determine the coefficients:
- $u(0) = u_1 = a_1$
- $u(L) = u_2 = a_2 L + u_1$
- $a_2 = \frac{u_2 - u_1}{L}$
- Therefore, expressed in nodal terms the displacement function becomes:
- $u = \left(\frac{u_2 - u_1}{L}\right) x + u_1$

Step 2: Displacement Function (Continued)

- Expressing in matrix form:
- $u = \begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$
- Or, alternatively;
- $u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$
- where,
- $N_1 = 1 - \frac{x}{L}$
- $N_2 = \frac{x}{L}$
- These are called the 'shape functions'
- $N_1 + N_2 = 1$

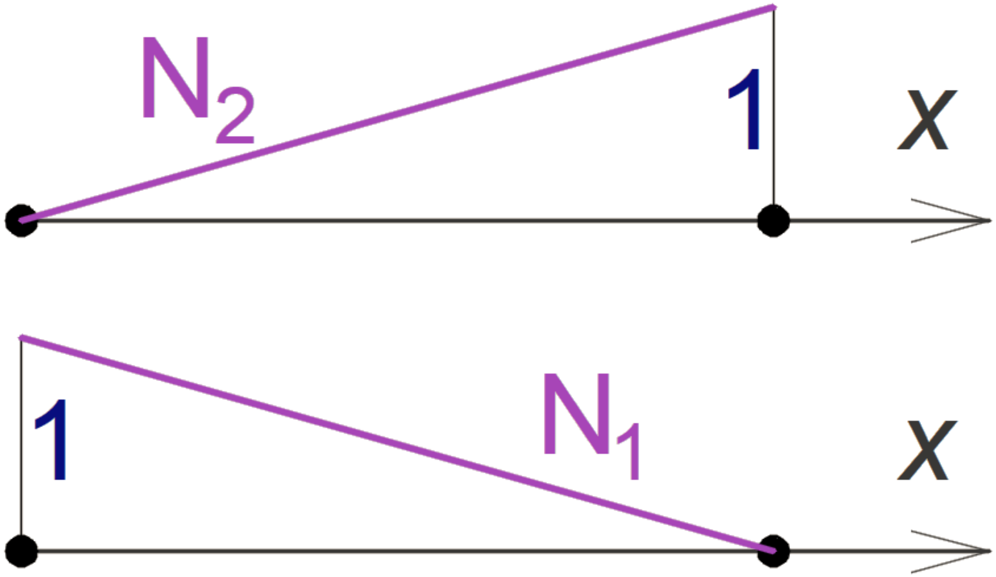


Figure 5

Step 3: Strain-Displacement and Stress-Strain Relationships

- Strain-Displacement Relationship:
- Total element displacement (change in length) can be expressed mathematically as the displacement of the end-point minus the displacement of the starting point:
- $\delta = u(L) - u(0) = u_2 - u_1$
- Assuming constant strain throughout the element:
- $\varepsilon = \frac{\delta}{L} = \frac{u_2 - u_1}{L}$

Step 3: Strain-Displacement and Stress-Strain Relationships (Continued)

- Stress-Strain Relationship:
- For a spring element, this relationship can be expressed in terms of Force/Deformation
- $T = k\delta$
- $T = k(u_2 - u_1)$
- Where, T is the force in the spring taken as positive if tensile

Step 4: Element Stiffness Equations and Matrix

- Considering the force in the spring to be positive if tensile and its value being T:
- $f_1 = -T; \quad f_2 = T$
- $T = -f_1 = k(u_2 - u_1)$
- $T = f_2 = k(u_2 - u_1)$
- $f_1 = k(u_1 - u_2)$
- $f_2 = k(u_2 - u_1)$
- $\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

Step 4: Element Stiffness Equations and Matrix (Continued)

- This is the system of equations that represents a spring element
- The element stiffness matrix can be extracted from this system of equations as:
- $[k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$
- It is the local stiffness matrix. Note that it is a symmetric square matrix
- Since both the springs in the example structure are the same, we can use this stiffness matrix for the both of elements

Step 4: Element Stiffness Equations and Matrix (Continued)

- $$\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} = \begin{bmatrix} 21 & -21 \\ -21 & 21 \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{Bmatrix}$$
- $$\begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} = \begin{bmatrix} 21 & -21 \\ -21 & 21 \end{bmatrix} \begin{Bmatrix} u_2^{(2)} \\ u_3^{(2)} \end{Bmatrix}$$
- Where, the superscripts (1) and (2) represent the element numbers
- These can be expressed as element equations as:
- $f_1^{(1)} = 21u_1^{(1)} - 21u_2^{(1)}$
- $f_2^{(1)} = -21u_1^{(1)} + 21u_2^{(1)}$
- $f_2^{(2)} = 21u_2^{(2)} - 21u_3^{(2)}$
- $f_3^{(2)} = -21u_2^{(2)} + 21u_3^{(2)}$

Step 5(a): Assembly of Structural Stiffness Matrix

- The structure has 3 nodes
- Assuming the global forces at these nodes to be F_1, F_2, F_3:
- To satisfy equilibrium:
- $F_1 = f_1^{(1)} = 21u_1^{(1)} - 21u_2^{(1)}$
- $F_2 = \left(f_2^{(1)} + f_2^{(2)}\right)$
- $F_2 = \left(-21u_1^{(1)} + 21u_2^{(1)} + 21u_2^{(2)} - 21u_3^{(2)}\right)$

Step 5(a): Assembly of Structural Stiffness Matrix (Continued)

- $F_3 = f_3^{(2)} = -21u_2^{(2)} + 21u_3^{(2)}$
- Compatibility requires that:
- $u_2^{(1)} = u_2^{(2)} = u_2; u_3^{(3)} = u_3; u_1^{(1)} = u_1$
- Therefore:
- $F_1 = 21u_1 - 21u_2$
- $F_2 = (-21u_1 + 21u_2 + 21u_2 - 21u_3) = -21u_1 + 42u_2 - 21u_3$
- $F_3 = -21u_2 + 21u_3$

Step 5(a): Assembly of Structural Stiffness Matrix (Continued)

- Re-arranging the element equations:
- $F_1 = 21u_1 - 21u_2 + 0u_3$
- $F_2 = -21u_1 + 42u_2 - 21u_3$
- $F_3 = 0u_1 - 21u_2 + 21u_3$
- Matrix Form:
- $\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} 21 & -21 & 0 \\ -21 & 42 & -21 \\ 0 & -21 & 21 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$

Step 5(a): Assembly of Structural Stiffness Matrix (Continued)

- Compact form:
- $\{F\} = [K]\{d\}$; where:
- $\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$; $\{d\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$; and
- $[K] = \begin{bmatrix} 21 & -21 & 0 \\ -21 & 42 & -21 \\ 0 & -21 & 21 \end{bmatrix}$
- This is the assembled stiffness matrix for the structure

Step 5(a): Alternative Method (Direct Stiffness)

- Element stiffness matrices can be written with rows and columns labelled with the corresponding degrees of freedom (DOF)

- $$\left[k^{(1)}\right] = \begin{array}{cc} u_1 & u_2 \\ \left[\begin{array}{cc} 21 & -21 \\ -21 & 21 \end{array}\right] \end{array} \begin{array}{c} \\ u_1 \\ u_2 \end{array}$$

- $$\left[k^{(2)}\right] = \begin{array}{cc} u_2 & u_3 \\ \left[\begin{array}{cc} 21 & -21 \\ -21 & 21 \end{array}\right] \end{array} \begin{array}{c} \\ u_2 \\ u_3 \end{array}$$

- These can be expanded as:

- $$\left[k^{(1)}\right] = \begin{array}{ccc} u_1 & u_2 & u_3 \\ \left[\begin{array}{ccc} 21 & -21 & 0 \\ -21 & 21 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array}$$

- $$\left[k^{(2)}\right] = \begin{array}{ccc} u_1 & u_2 & u_3 \\ \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 21 & -21 \\ 0 & -21 & 21 \end{array}\right] \end{array} \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array}$$

- Adding the two:

- $$[K] = \begin{array}{ccc} u_1 & u_2 & u_3 \\ \left[\begin{array}{ccc} 21 & -21 & 0 \\ -21 & 42 & -21 \\ 0 & -21 & 21 \end{array}\right] \end{array} \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array}$$

Step 5(b): Boundary Conditions

- The system of equations for the structure is:
- $$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} 21 & -21 & 0 \\ -21 & 42 & -21 \\ 0 & -21 & 21 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$
- The stiffness matrix is singular i.e. this system has no solution
- To get a solution we need to apply the boundary conditions
- The known boundary condition is:
- $u_1 = 0$, since node 1 is fixed
- Also, we know that: $F_2 = F_3 = 6kN$

Step 5(b): Boundary Conditions (Continued)

- Applying this condition:
- $\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} 21 & -21 & 0 \\ -21 & 42 & -21 \\ 0 & -21 & 21 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix}$
- $F_1 = -21u_2$
- $6 = 42u_2 - 21u_3$
- $6 = -21u_2 + 21u_3$
- The last 2 equations can be expressed in matrix form as:
- $\begin{Bmatrix} 6 \\ 6 \end{Bmatrix} = \begin{bmatrix} 42 & -21 \\ -21 & 21 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$
- Note that homogeneous boundary conditions can be directly applied by removing the rows and columns corresponding to the zero displacements from the system of equations

Step 6: Solution of the System of Equations

- The solution of this system of equations is straight forward
- $u_2 = 0.57m$
- $u_3 = 0.86m$
- These are the structural displacements. The displacements for the individual elements can be expressed using compatibility:
- $u_1^{(1)} = u_1 = 0$
- $u_2^{(1)} = u_2^{(2)} = 0.57m$
- $u_3^{(3)} = 0.86m$

Step 7: Element Deformation and Forces

- For spring elements we are using deformations and forces instead of strains and stresses
- $\delta^{(1)} = u_2 - u_1 = 0.57 - 0 = 0.57m$ (elongation)
- $\delta^{(1)} = u_3 - u_2 = 0.86 - 0.57 = 0.29m$ (elongation)
- Tension in each of the elements can be determined as:
- $T = k\delta$
- $T^{(1)} = 21 \times 0.57 = 11.97\ kN \cong 12kN$
- $T^{(2)} = 21 \times 0.29 = 6.09\ kN \cong 6kN$

Step 8: Interpretation of Results

- In the previous step, we obtained the deformation and the tension in each of the elements
- Interpretation of these results is not directly related to Finite Element Analysis
- Whether the deformations are within the allowable limits and whether the tension would cause the material to fail is up to the engineer to decide
- Some of the students might have noticed that the tension obtained in the elements is only approximately equal to the values suggested by simple equilibrium
- This is because only 2 significant digits after the decimal were considered while solving the system of equations
- Increasing the number of significant digits will bring the solution closer to the exact solution, however, it will be exactly the same only if the computing machine being used has infinite precision

Author Information

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