Universiti Malaysia PAHANG
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NUMERICAL METHODS & OPTIMISATION

Part I: Optimisation

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Chapter Description

- Aims
  - Apply numerical methods in solving engineering problem and optimisation

- Expected Outcomes
  - Calculate the maximum, minimum and optimal values of an equation by using the following methods for optimisation:
    - Golden-Section Search
    - Quadratic Interpolation
  - Solve engineering problems by using methods for optimisation

- References
  - Steven C. Chapra and Raymond P. Canale (2009), Numerical Methods for Engineers, McGraw-Hill, 6th Edition

Overview of Optimization

- Root of equation (Chapter 2) and optimization (Chapter 4) are similar
  - Root of equation: finding the values of x at $f(x) = 0$
  - Optimization: finding the values of x at maximum and minimum point of $f(x)$
- Optimum is the point where the curve is flat – values of $f'(x) = 0$
- The second derivatives $f''(x)$ indicates whether the point at which slope is zero is minimum or maximum
  - $f''(x) < 0$: maximum point
  - $f''(x) > 0$: minimum point
- These values are known as optima
  - $f(x) = 0$ – root
  - $f'(x) = 0$ – optima

Overview of Optimization (cont'd)

- Understanding the relationship between root and optima helps to determine the suitable method to be used in finding optima
- Example:
  - Through the differentiation of a function f'(x) and later locate the root of the newly derived function
  - Complex – f'(x) is sometime not available analytically
  - Different strategy should be used: e.g. finite difference approximations to estimate the derivatives
- Optimization deals with finding the best results or optimum solution using the prescriptive models

Optimization: Case study

- Optimization of Parachute Cost
- Problem statement:
  - To determine the size)r) and number of chutes (n) that result in the minimum cost while at the same time having the small impact velocity

$$Cost\ per\ chute = c_0 + c_1 l + c_2 A^2$$

- Information
  - Supplies will be dropped at an altitude of 500m
  - The chute opens immediately upon leaving the plane
  - Vertical velocity of impact should be below v_c = 20 m/s
  - c_o is the base price for the chutes
  - The relationship is non-linear – larger chutes are difficulte to construct thus the price is higher

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Optimization: Case study (cont'd)

- Given:

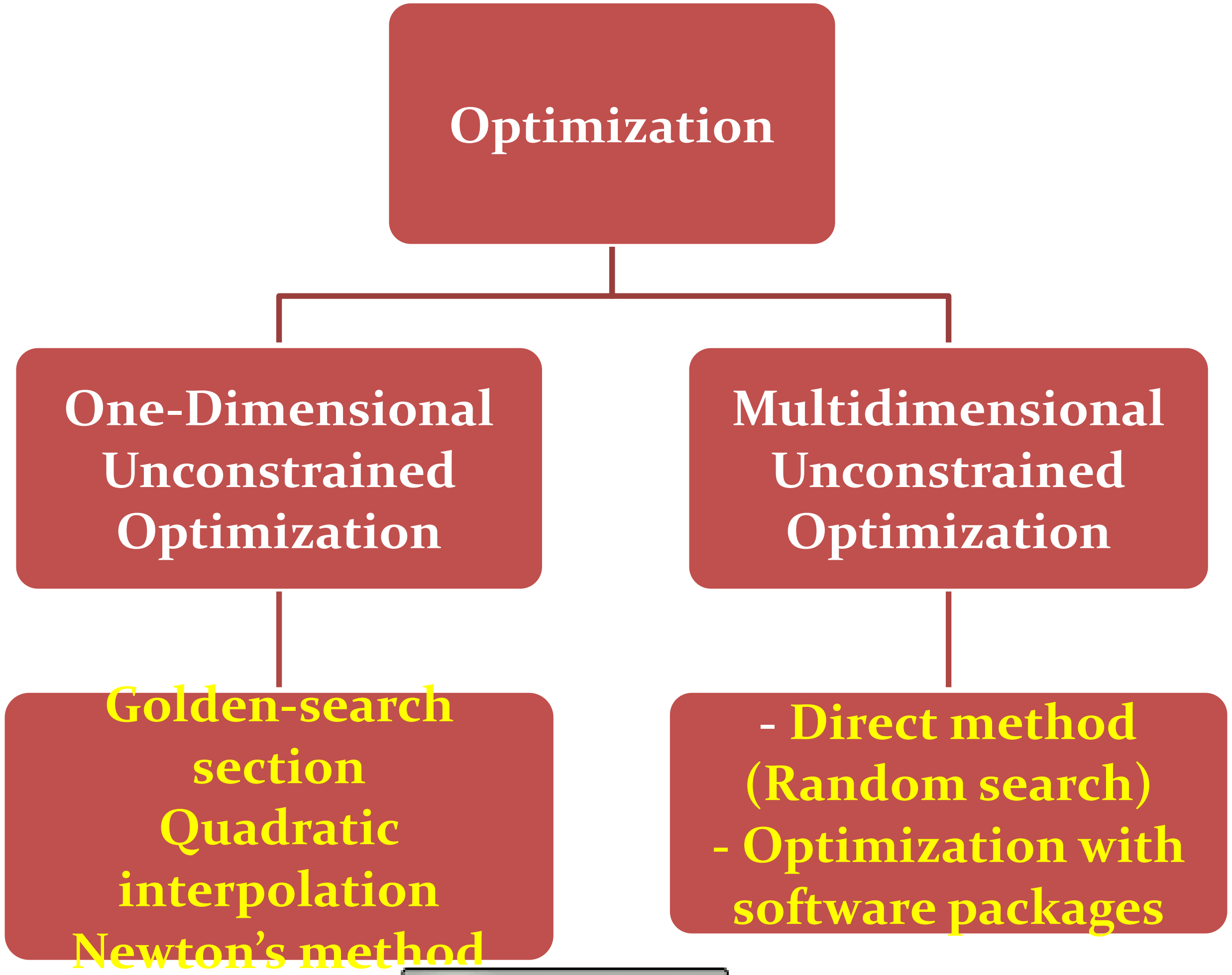
$A = 2\pi r^2$: Cross sectional area of balloon

$l = \sqrt{2r}$: Length connecting chute & mass

$c = k_c A$: Drag coefficient; effect of area on drag coefficient

$m = \frac{M_t}{n}$: Mass of individual parcel

Overview of Optimization (cont'd)



One – dimensional Unconstrained Optimization

- Root of equation – several roots can occur for a single function – multimodal
- To find the min and maximum value of a single variable function, f(x)
- These values are known as optima – local & global optima – difficult to be distinguished
- 3 approaches:
  - Graphical method
  - Random initial guesses – the largest value is the global optima
  - Perturbing the starting point associated with a local optimum – checking whether the routine returns a better point or always return to the same point
- One dimension optimization involves 2 types of methods:
  - Bracketing methods: Golden-search section, Quadratic interpolation
  - Open methods: Newton method

Golden-section Search

- Single variable optimization has the goal of finding the value of x that yields an extremum – maximum & minimum of f(x)
- Simple, general purpose & similar to bisection method with some modifications to find minimum
- The initial step of involves a selection of two interior points according to the **golden ratio**.
- Golden-search calculation involves a selection of 4 values: 2 extrems of interval: x_{upper} and x_{lower}, 2 intermediate values, x_1 & x_2

Golden-section search (cont'd)

- The first condition specifies that the sum of the two sub lengths l_1 and l_2 must be equal to the original interval length.
- The second condition assumes that the ratio of the length must be equal.
- The R value is called Golden Ratio which allows the optima to be found efficiently

$$l_0 = l_1 + l_2$$

$$\frac{l_1}{l_0} = \frac{l_2}{l_1}$$

$$\frac{l_1}{l_1 + l_2} = \frac{l_2}{l_1} \qquad R = \frac{l_2}{l_1}$$

$$1 + R = \frac{1}{R} \qquad R^2 + R - 1 = 0$$

$$R = \frac{-1 + \sqrt{1 - 4(-1)}}{2} = \frac{\sqrt{5} - 1}{2} = 0.61803$$

Golden-section search: Algorithm

- Select two initial guesses, x_l and x_u that bracket one local extremum
- Determine two intermediate values, x1 & x2 from the following equations. These values are chosen according to the golden ratio

$$d = \frac{\sqrt{5}-1}{2}(x_u - x_l) \qquad x_1 = x_l + d \qquad x_2 = x_u - d$$

- Test the results for the following conditions:
  - If $f(x_1) > f(x_2)$: x is to the left of x_2 from x_l to x_2, thus x_2 is the new x_l
  - If $f(x_2) > f(x_1)$: x is to the right of x_2 from x_1 to x_u, thus x_2 is the new x_u
- Determine the approximation error:

$$\varepsilon_a = (1-R)\left|\frac{x_u - x_l}{x_{opt}}\right| 100\%$$

Golden-section search: Example

Use the golden-section search to find the maximum of

$$f(x) = 2\sin x - \frac{x^2}{10}$$

within the interval $x_l = 0$ and $x_u = 4$

and **perform three iterations**. Calculate ε_a after each iteration.

Quadratic Interpolation

- Involves 3 points connected by one quadratic function

$$x_3 = \frac{f(x_0)(x_1^2 - x_2^2) + f(x_1)(x_2^2 - x_0^2) + f(x_2)(x_0^2 - x_1^2)}{2f(x_0)(x_1 - x_2) + 2f(x_1)(x_2 - x_0) + 2f(x_2)(x_0 - x_1)}$$

- Where x_0, x_1 *and* x_2 are the initial guesses and x_3 is the value of *x* that corresponds to the maximum value of the quadratic fit to the guesses.
- Strategy is similar to golden-section search to decide which point to be discarded

Quadratic Interpolation: Example

Use quadratic interpolation to approximate the maximum of

$$f(x) = 2\sin x - \frac{x^2}{10}$$

within initial guesses of $x_0 = 0$, $x_1 = 1$ and $x_2 = 4$, and **perform three iterations**.

Exercise

a) Solve for the value of x that maximizes *f(x)* in Prob 13.2 using golden section search. Employ initial guesses of $x_l = 0$ **and** $x_u = 2$ and **perform three iterations**.

$$f(x) = -1.5x^6 - 2x^4 + 12x$$

b) Repeat (a), except use quadratic interpolation. Employ initial guesses of $x_0 = 0$, $x_1 = 1$ ***and*** $x_2 = 2$, and **perform three iterations**.

Conclusion

- The maximum, minimum and optimal values of an equation can be estimated by using Golden-Section Search and Quadratic Interpolation for optimization

Main Reference

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