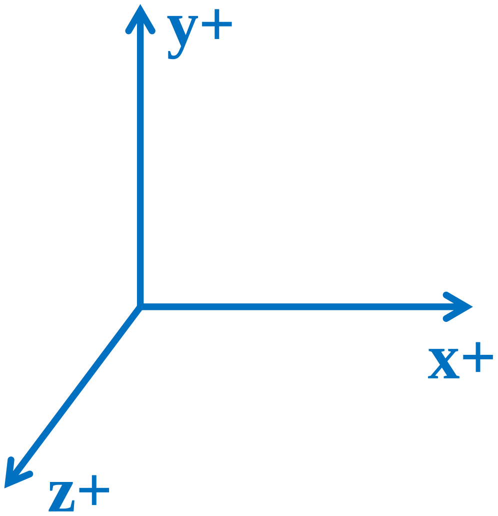
COMPUTER AIDED ENGINEERING DESIGN (BFF2612)

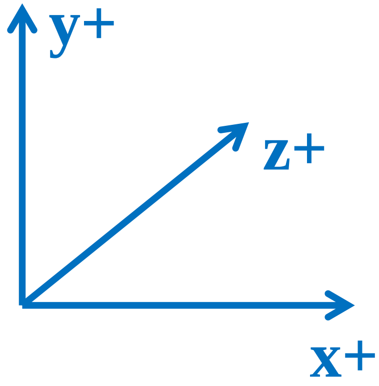
BASIC MATHEMATICAL CONCEPTS IN CAED

by
Dr. Mohd Nizar Mhd Razali
Faculty of Manufacturing Engineering
mnizar@ump.edu.my

COORDINATE SYSTEM



RIGHT HANDED
Coordinate System



LEFT HANDED
Coordinate System

We adopt RIGHT HANDED Coordinate System.

PARAMETRIC SYSTEM

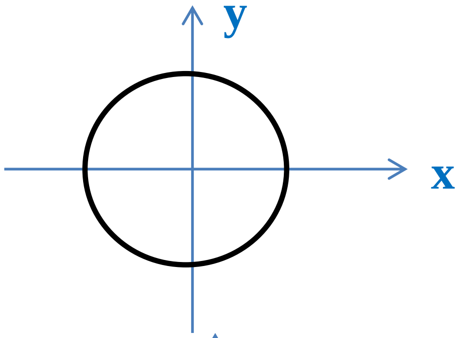
Most CAD software use parametric system. Parametric curves:

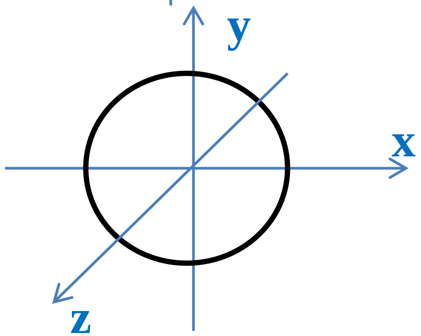
- Each coordinate of a point on a curve is represented as a function of one or more parameters.
- The position vector of a point on the curve is fixed by the value of the parameter.
- It is well suited for computations and display.

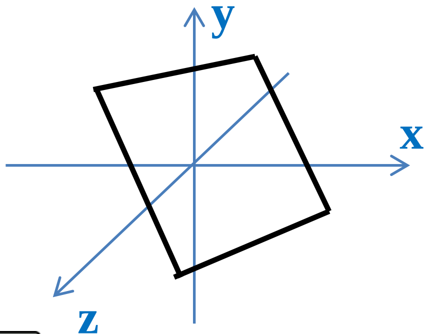
Limitations of non-parametric representation:

1. If the slope of a curve at a point is vertical, its value becomes infinity. It is difficult for computation and program (y = mx + c).
2. Shapes of engineering objects are independent to any coordinate system.
3. If the curve is to be displayed as a series of points, straight line segments, the computations is extensive.

Representations of Geometry (2D and 3D Shapes)







Non-Parametric Representations:

Circle (2D):

$x^2 + y^2 = R^2$ Implicit

$y = \pm\sqrt{R^2 - x^2}$ Explicit

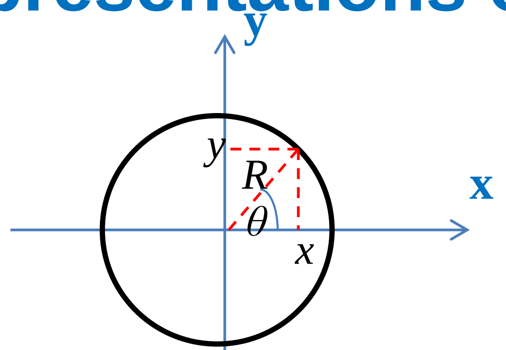
Sphere (3D):

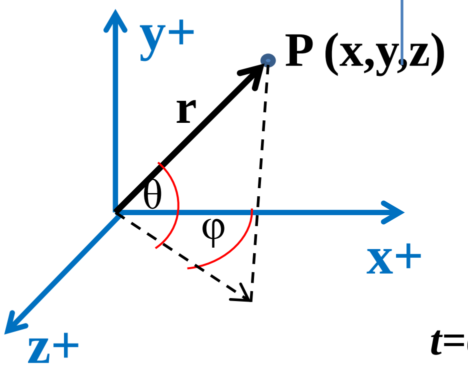
$$x^2 + y^2 + z^2 = R^2$$

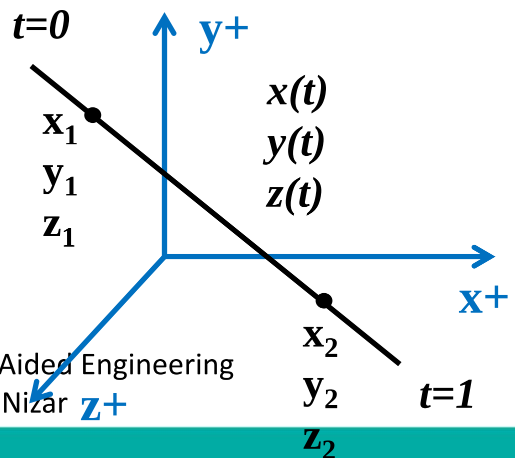
Plane (3D):

$$Ax + By + Cz = D$$

Representations of Geometry (2D and 3D Shapes)







Parametric representations:

Circle

$$x = R\cos\theta$$
$$y = R\sin\theta$$
$$0 \leq \theta \leq 2\pi$$

Sphere

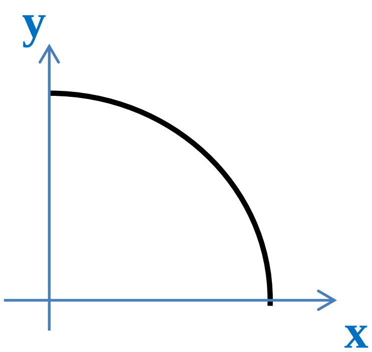
$$x = r\cos\emptyset\cos\theta$$
$$y = r\cos\emptyset\sin\theta$$
$$z = r\cos\emptyset$$
$$-\frac{\pi}{2} \leq \emptyset \leq \frac{\pi}{2},\ 0 \leq \theta \leq 2\pi$$

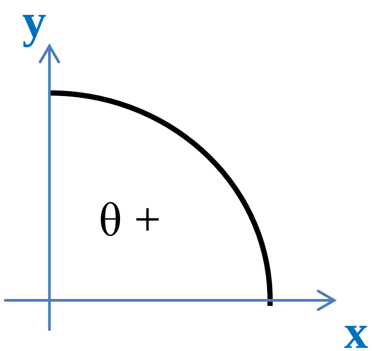
Line

$$x(t) = x_1 + (x_2 - x_1)t$$
$$y(t) = y_1 + (y_2 - y_1)t$$
$$z(t) = z_1 + (z_2 - z_1)t$$

CALCULATION EXAMPLE OF PARAMETRIC AND NON PARAMETRIC

Comparison of non parametric and parametric representations for a circle in the first quadrant





- Non parametric representation for unit circle in first quadrant (R=1):

$$y = +\sqrt{1 - x^2} \qquad 0 \leq x \leq 1$$

- Parametric form for unit circle

$$x = \cos\theta$$

$$y = \sin\theta$$

$$P(\theta) = [x \quad y] = [\cos\theta \quad \sin\theta]$$

$$0 \leq \theta \leq 2\pi$$

CALCULATION EXAMPLE OF PARAMETRIC AND NON PARAMETRIC

Determination of a point on a parametric curve.

Determine the value of y for a given x

e.g. $x=0.5$, $R=1$ for unit circle

Explicit non parametric representation

$$y = \sqrt{1 - x^2} = \sqrt{1 - 0.5^2} = \sqrt{0.75} = 0.866$$

Parametric representation

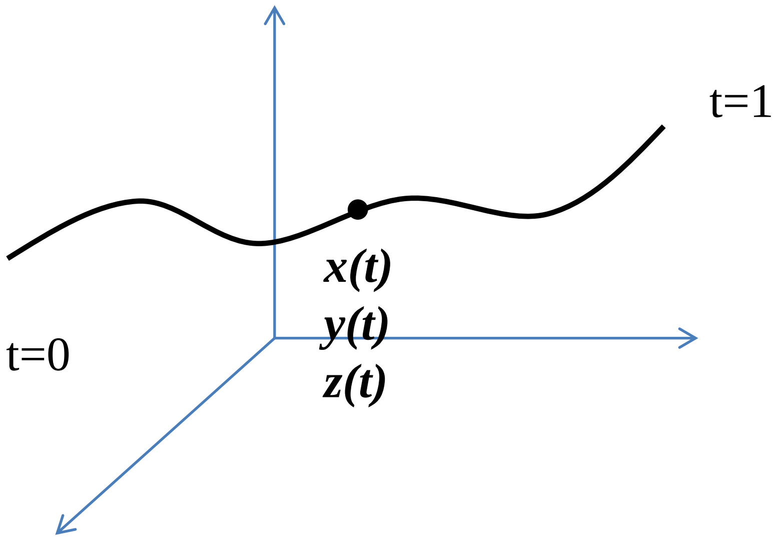
$$x = \cos\theta$$
$$y = \sin\theta$$

First solve for the parameter θ in x

$$\theta = \cos^{-1}(x) = \cos^{-1}(0.5) = 60^0$$
$$y = \sin(60^0) = 0.866$$

For more complex parametric representations => iterative techniques

Parametric Expression of Curve



Curve

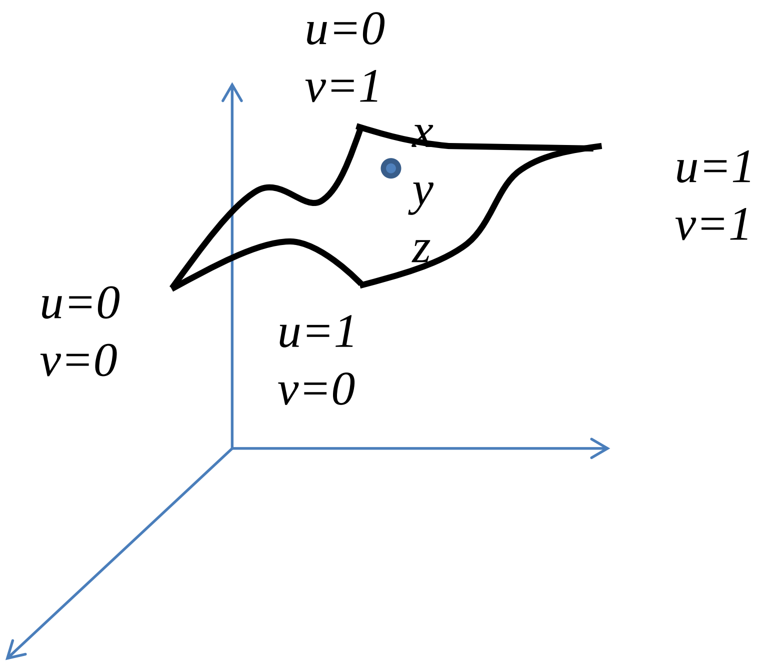
$$x(t) = f_x(t)$$
$$y(t) = f_y(t)$$
$$z(t) = f_z(t)$$

Functions of 1 parameter *t*

$$0 \le t \le 1$$

Example:

Parametric Expression of Surface

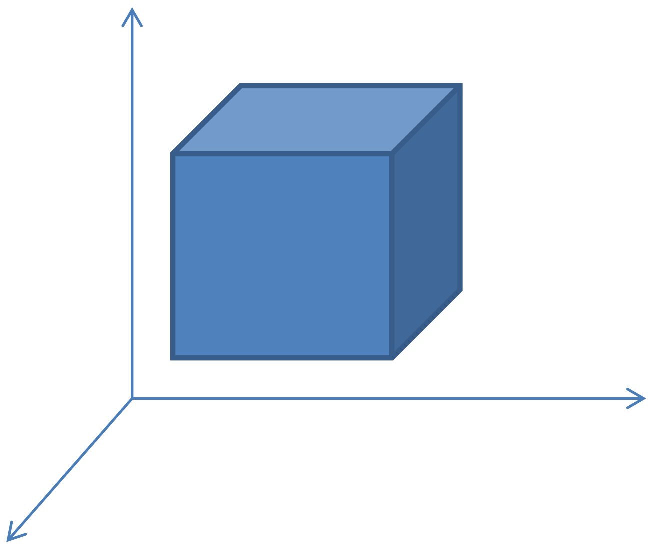


Surface

$$x(u,v) = f_x(u,v)$$
$$y(u,v) = f_y(u,v)$$
$$z(u,v) = f_z(u,v)$$

Functions of 2 parameters u, v

Parametric Expression of Solid



Solid

$$x(u,v,w) = f_x(u,v,w)$$
$$y(u,v,w) = f_y(u,v,w)$$
$$z(u,v,w) = f_z(u,v,w)$$

Functions of 3 parameters u, v, w

CURVE CATEGORIES

Two categories of curves represented parametrically:

1. Analytic. Described by analytical equations. Example lines, circles, conics.

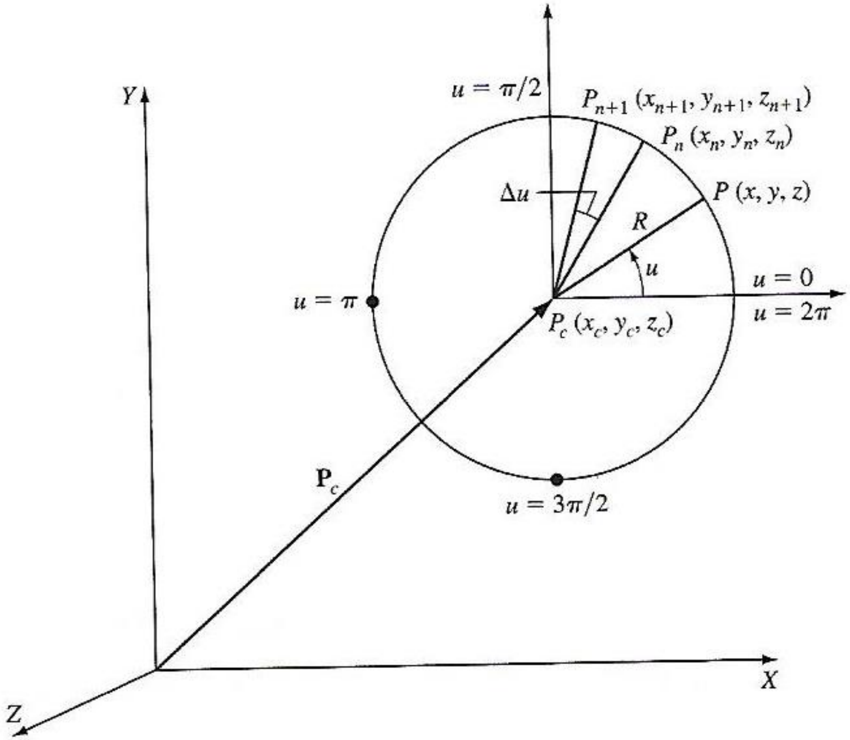
2. Synthetic. Described by sets of data points, such as splines and Bezier curves. Provide greater flexibility and control of a curve shape by changing the positions of the control points. Especially applied for curves and surfaces.

ANALYTIC CURVE

Circle

$$x = x_c + R\cos\theta$$
$$y = y_c + R\sin\theta$$
$$z = z_c$$

Centre of circle is $(x_c y_c z_c)$, radius is R and $0 \le \theta \le 2\pi$

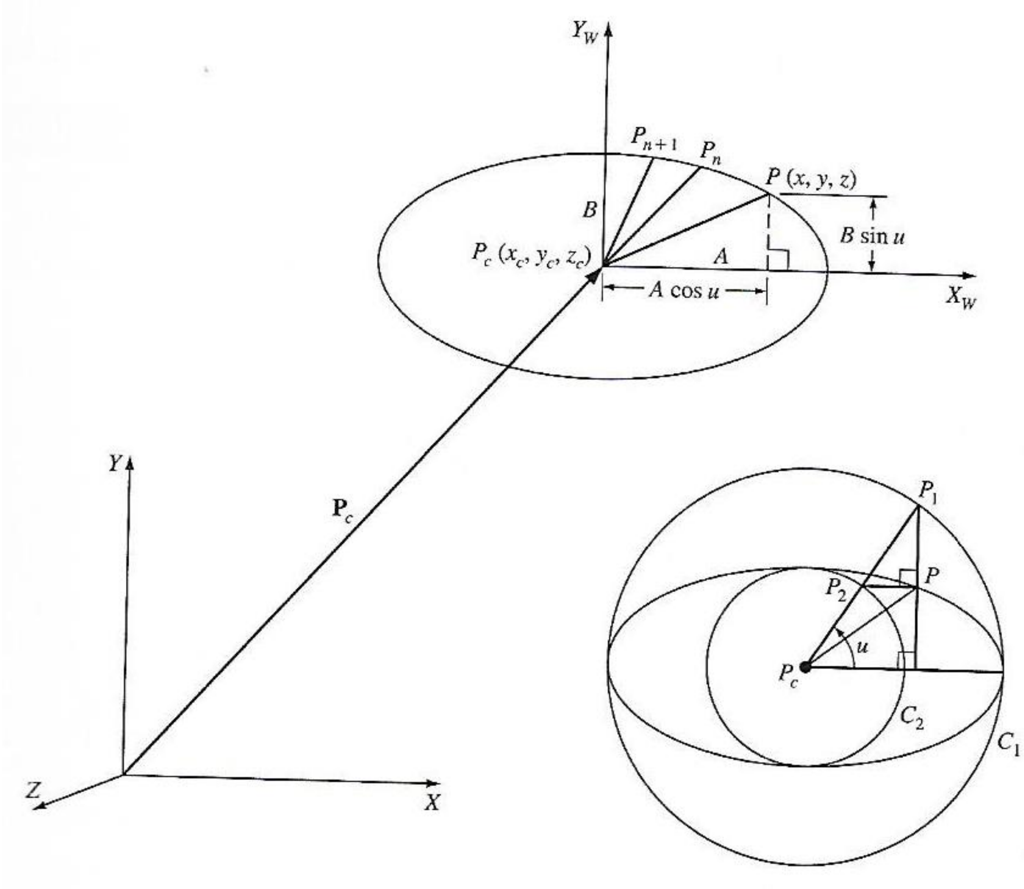


ANALYTIC CURVE

Ellipse

$$x = x_c + A\cos\theta$$
$$y = y_c + B\sin\theta$$
$$z = z_c$$

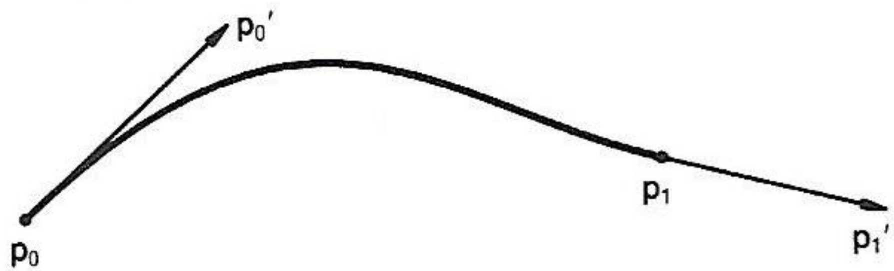
Centre of ellipse is $(x_c y_c z_c)$ and $0 \le \theta \le 2\pi$



SYNTHETIC CURVE

- Parametric cubic polynomial curves (Hermite curves)
- Bezier curves
- Cubic spline curves
- B-spline curves

Parametric cubic polynomial curves (Hermite curves)



A cubic curve may equally well be defined to fit two points and two slope conditions at the points.

Parametric cubic polynomial curves (Hermite curves)

The parametric equation of a cubic spline segment is given by:

$P(u) = \sum_{i=0}^{3} C_i u^i \qquad 0 \leq u \leq 1$

Where u is the parameter and C_i are the polynomial (also called algebraic) coefficients. In scalar form this equation is written as:

$$x(u) = C_{3x}u^3 + C_{2x}u^2 + C_{1x}u + C_{0x}$$
$$y(u) = C_{3y}u^3 + C_{2y}u^2 + C_{1y}u + C_{0y}$$
$$z(u) = C_{3z}u^3 + C_{2z}u^2 + C_{1z}u + C_{0z}$$

We have 12 unknowns and we can use Lagrange interpolation to solve these equations with four points.

In vector form:

$P(u) = C_3u^3 + C_2u^2 + C_1u + C_0 \qquad$ (a)

The tangent vector (slope) of the curve is:

$P'(u) = 3C_3u^2 + 2C_2u + C_1 \qquad$ (b)

Parametric cubic polynomial curves (Hermite curves)

Using the end points p_0 and p_1, and the end slopes $p_0{}'$ and $p_1{}'$, we can substitute in Equations (a) and (b) to derive the unknowns. It is usual to assign $u = 0$ and $u = 1$ to the two ends of the segment. Thus:

$$p_0 = C_0$$
$$p_0{}' = C_1$$
$$p_1 = C_3 + C_2 + C_1 + C_0$$
$$p_1{}' = 3C_3 + 2C_2 + C_1$$

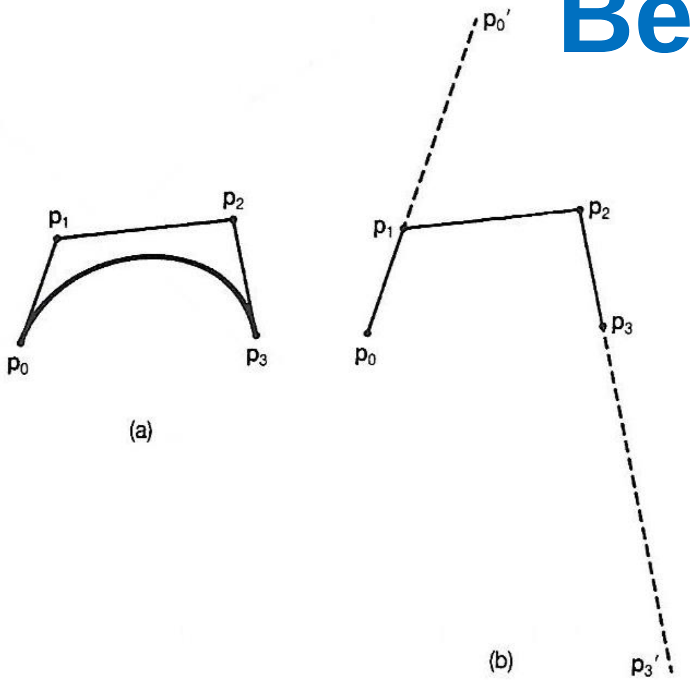
Solving for C_0 to C_3 we obtain:

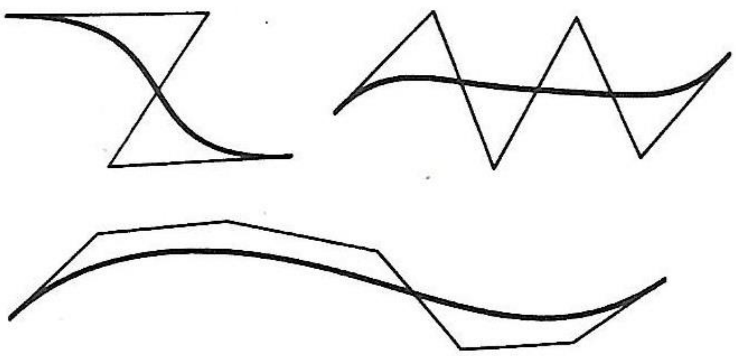
$$C_0 = p_0$$
$$C_1 = p_0{}'$$
$$C_2 = 3(p_1 - p_0) - 2p_0{}' - p_1{}'$$
$$C_3 = 2(p_0 - p_1) + p_0{}' + p_1{}'$$

Thus by substitution in Equation (a), we obtain (general form of a Hermit cubic polynomial:

$$P(u) = p_0(1 - 3u^2 + 2u^3) + p_1(3u^2 - 2u^3) + p_0{}'(u - 2u^2 + u^3) + p_1{}'(-u^2 + u^3)$$

Bezier Curves





- The curves controlled by its defining points, the curve is always tangent to the first and last polygon segments.
- The use of points and tangent vectors to provide boundary values for curves is not attractive for interactive design, because the user may not have much feel for the slopes entered as numerical values.

Bezier Curves

Bezier used a control polygon for curves. The mid vertices (control points) are defined to be 1/3 of the way along the tangent vectors.

Hermite curves	***Bezier curves***
p_0	p_0
p_1	p_3

$$p_0{}' = 3(p_1 - p_0)$$
$$p_3{}' = 3(p_3 - p_2)$$

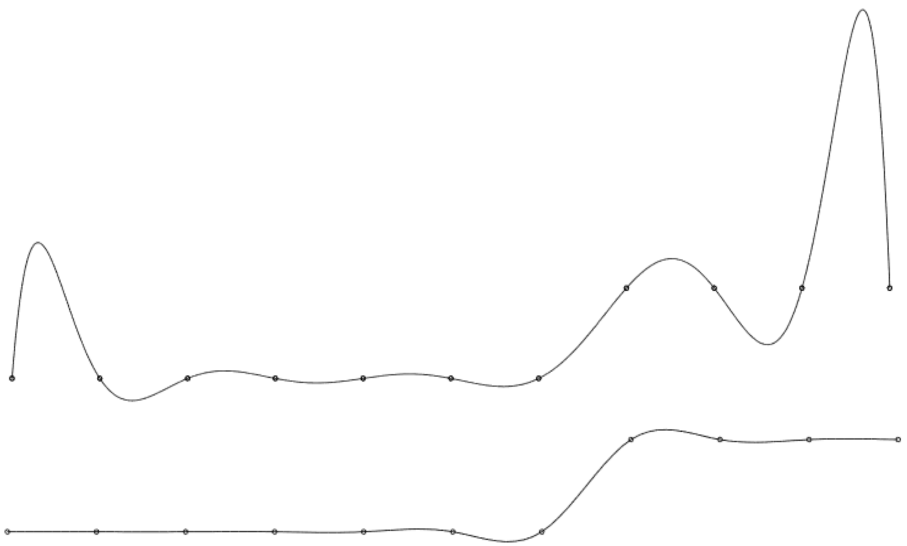
Substitute these into Equation below (Hermite curves) and gathering terms:

$$P(u) = p_0(1 - 3u^2 + 2u^3) + p_1(3u^2 - 2u^3) + p_0{}'(u - 2u^2 + u^3) + p_1{}'(-u^2 + u^3)$$

We obtain:

$$P(u) = p_0(1 - 3u + 3u^2 - u^3) + p_1(3u - 6u^2 + 3u^3) + p_2(3u^2 - 3u^3) + p_3(u^3)$$

Cubic Spline Curves



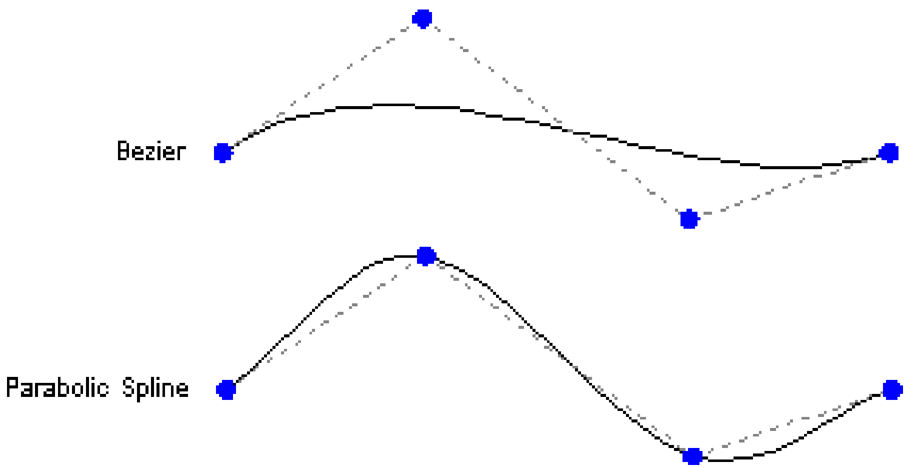
Normal interpolation of data points

Cubic spline interpolation of data points

- Curves are *piecewise* cubic curves, made of pieces of different cubic curves glued together.
- The pieces are so well matched where they are glued that the gluing is not obvious.

B-spline Curves

- Almost the same like Bezier curves (Generalization of the Bezier curves) only that it had second order continuity (1 degree more than Bezier).
- B-spline used control points for the curves.
- It is a continuous cubic polynomial that interpolates (passes through) the control points.
- It produces smoother curve. The polynomial coefficients depend on all central points.



<u>Example of Spline:</u>
Flexible strip of metal used by draft persons to lay out the surfaces of airplanes, cars & ships.

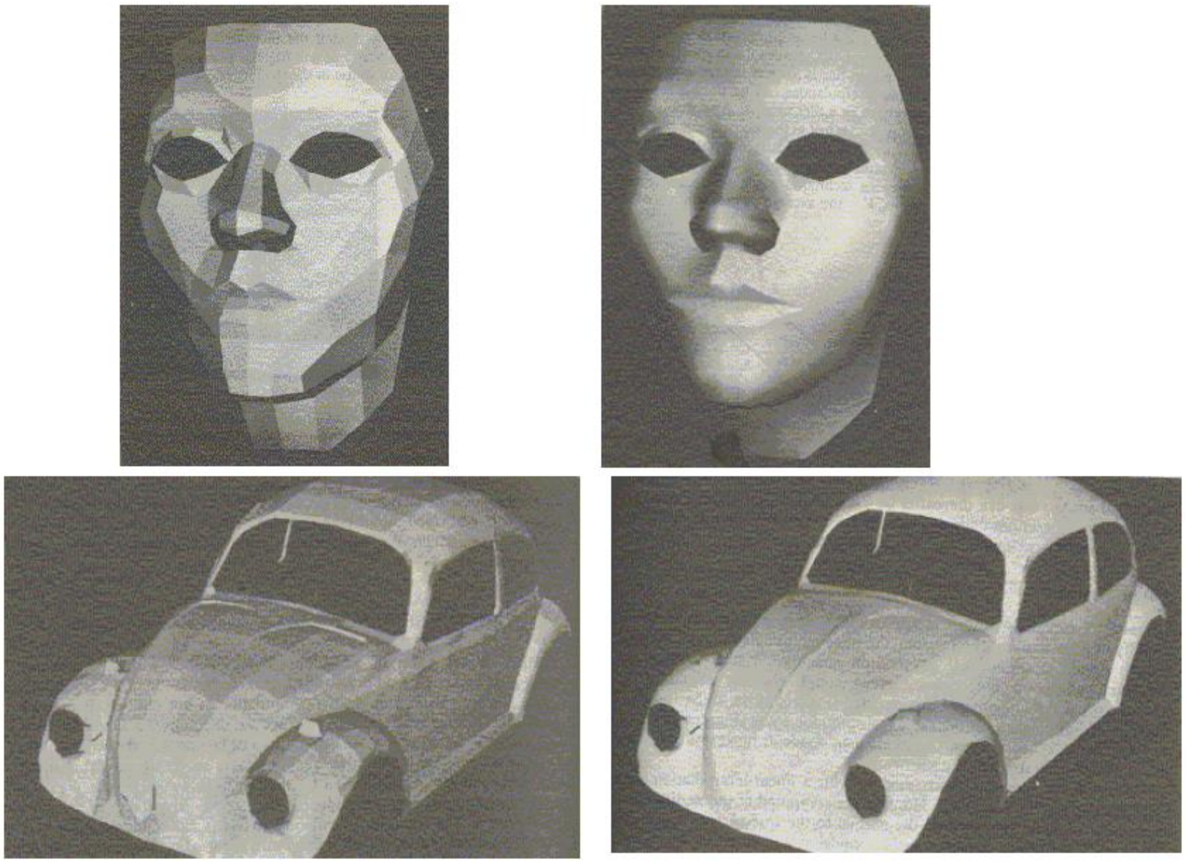
Representations of Geometry

Curve Representation

- • Points
- --- straight line segments
- — Curve

gives poor representation where radius of curvature is small

EXAMPLE



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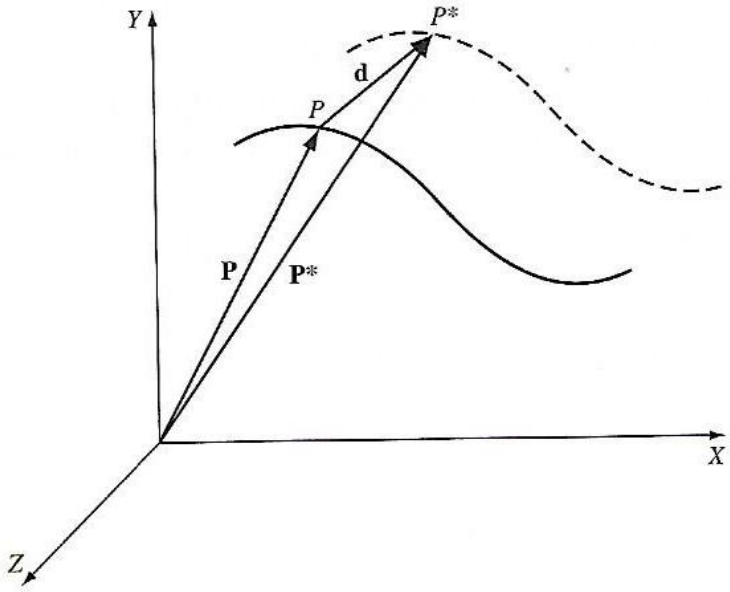
POLYGON MESH SMOOTH CURVES

OPERATION MATRICES

Translation

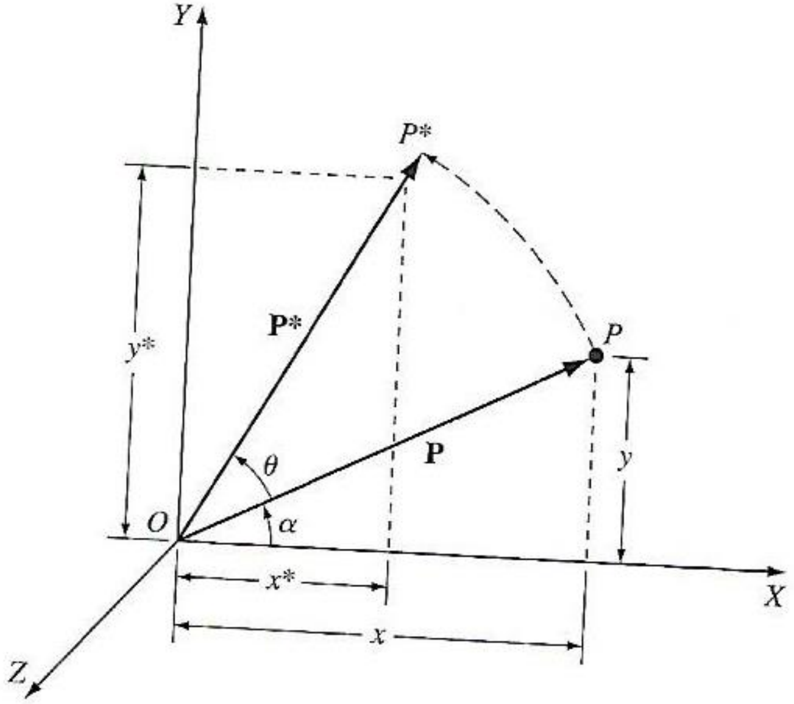


$$P^{*} = P + d$$

$$\begin{bmatrix} x^{*} \\ y^{*} \\ z^{*} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x_d \\ y_d \\ z_d \end{bmatrix}$$

OPERATION MATRICES

Rotation



$$P^* = [R\ \ ]P$$

About x axis

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

About y axis

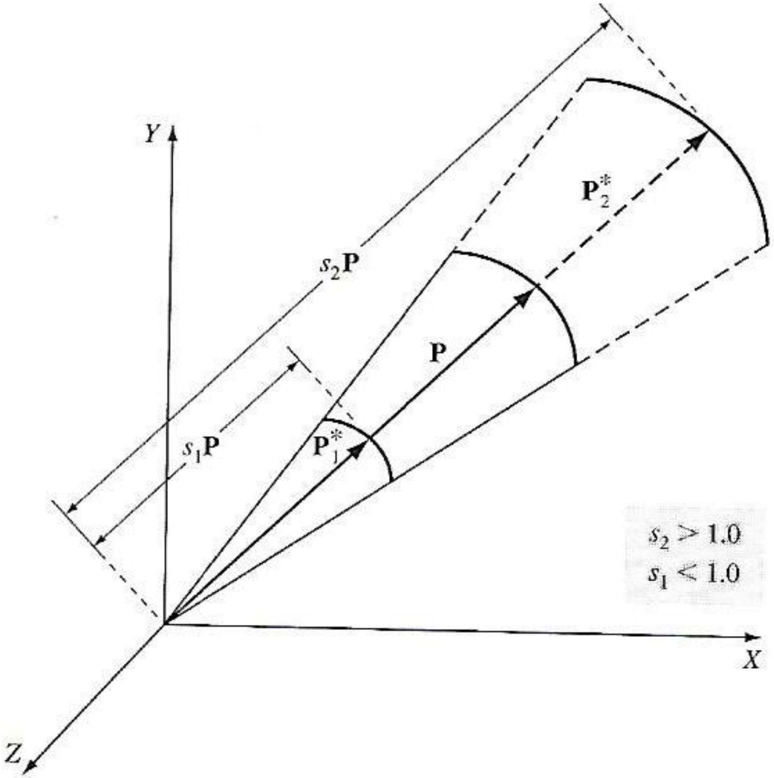
$$[R_y] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

About z axis

$$[R_z] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

OPERATION MATRICES

Scaling



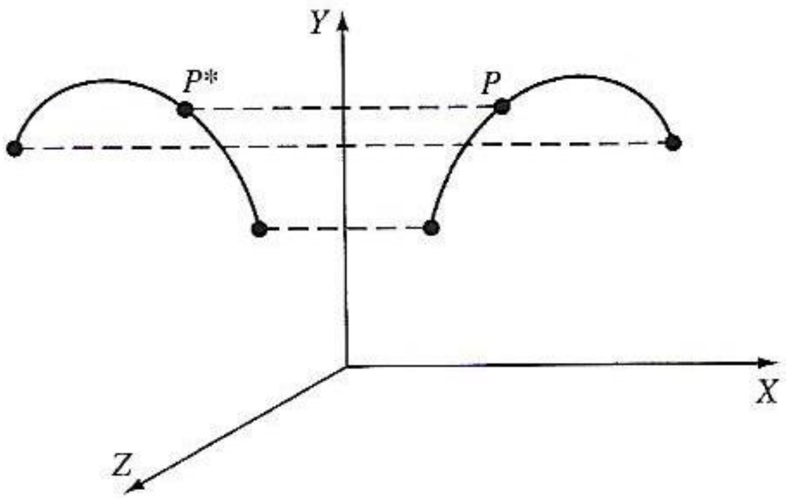
$$P^* = [S]P$$

$$\begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \begin{bmatrix} x_d \\ y_d \\ z_d \end{bmatrix}$$

OPERATION MATRICES

Reflection

$$P^* = [M]P$$



Reflection through a principal plane (x = 0 plane)

$$[M] = \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

Have any questions?

Thank you and Have a nice day!

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