DUM 2413 STATISTICS & PROBABILITY

# CHAPTER 5
# CONTINUOUS PROBABILITY DISTRIBUTIONS

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# EXPECTED OUTCOMES

- Able to determine the expected value, standard deviation and variance of a continuous random variable
- Able to identify the relationship between the normal distribution and the sampling distribution of the mean
- Able to solve the application problems, which involved the normal distribution and the sampling distribution of the mean
- Able to identify the relationship between the Binomial and Poisson distribution with a standard normal distribution

# CONTENT

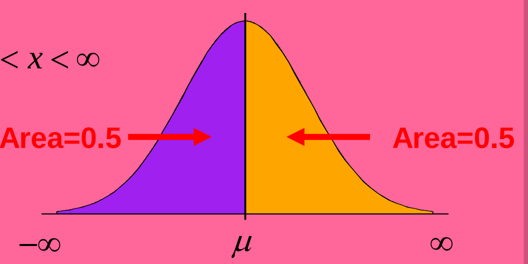
# 5.3
# NORMAL DISTRIBUTION

# NORMAL DISTRIBUTION vs. STANDARD NORMAL DISTRIBUTION

## NORMAL DISTRIBUTION

A ***continuous random variable,*** $X$ is said to follow ***normal distribution*** with parameters $\mu$ and $\sigma^2$ denoted by $X \sim N\left(x;\mu,\sigma^2\right)$ if

$$f\left(x\right)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\left(x-\mu\right)^2}{2\sigma^2}\right);\qquad -\infty<x<\infty$$

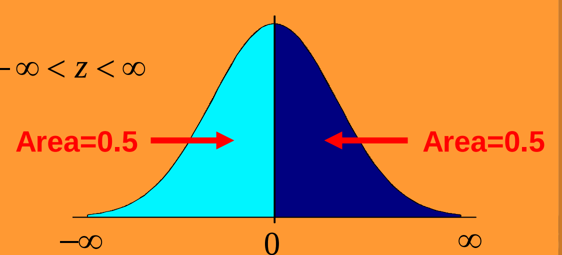



where the mean is $\mu$ and variance is $\sigma^2$.

## STANDARD NORMAL DISTRIBUTION

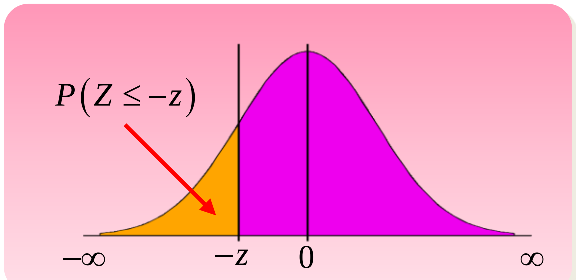
The normal distribution with $\mu=0$ and $\sigma=1$ is referred to as ***standard normal distribution,*** $Z \sim N\left(x;0,1\right)$

$$f\left(z\right)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right);\qquad -\infty<z<\infty$$

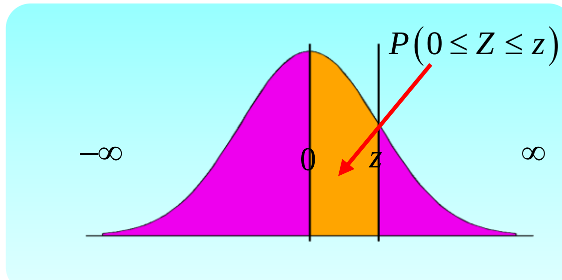



where the $z$ -score, $z=\frac{X-\mu}{\sigma}$

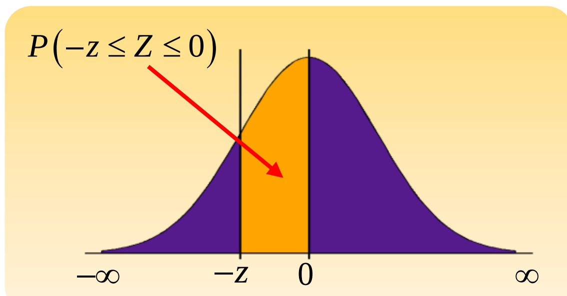
# FIND THE TOTAL AREA UNDER THE STANDARD NORMAL USING THE FUNCTIONS IN CALCULATOR



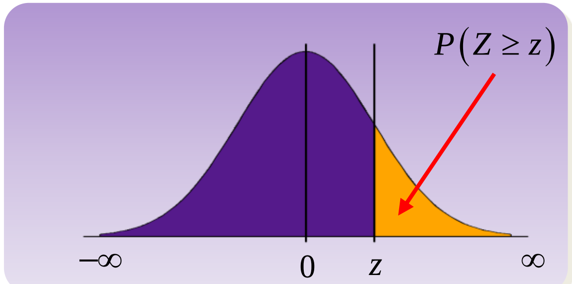

**Function In Calculator:** $\mathrm{P}(-z)$




**Function In Calculator:** $\mathrm{Q}(z)$




**Function In Calculator:** $\mathrm{Q}(-z)$




**Function In Calculator:** $\mathrm{R}(z)$

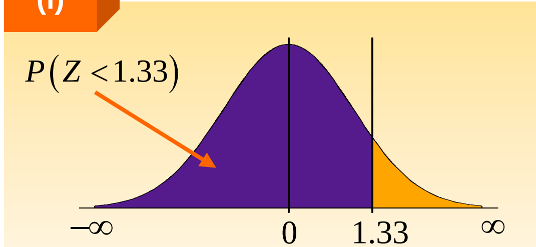
# EXAMPLE 5.6

Given that $Z$ is a continuous random variable follows the standard normal distribution. Find

(i) $P(Z<1.33)$. (ii) $P(Z\geq -0.79)$. (iii) $P(0.55<Z<1.22)$. (iv) $P(-1.90\leq Z\leq 0.44)$.

## SOLUTION

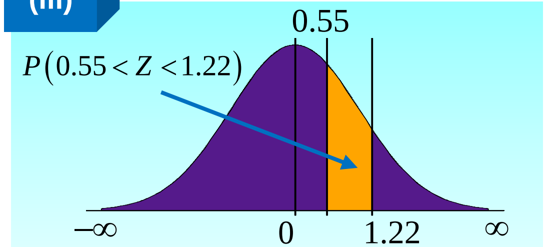
(i)




By using the function in calculator:

$$P(Z<1.33)=0.9082$$

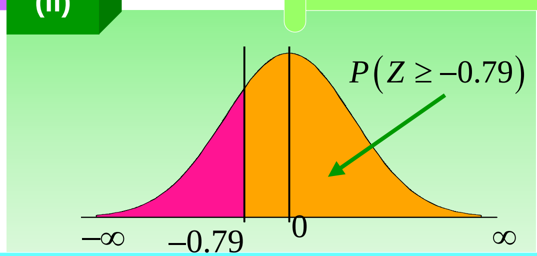
(iii)




By using the function in calculator:

$$P(0.55<Z<1.22)=0.1799$$

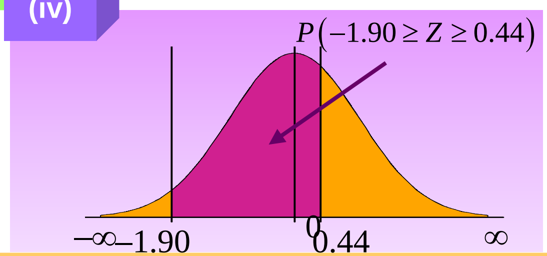
(ii)




By using the function in calculator:

$$P(Z\geq -0.79)=0.7852$$

(iv)




By using the function in calculator:

$$P(-1.90\geq Z\geq 0.44)=0.6413$$

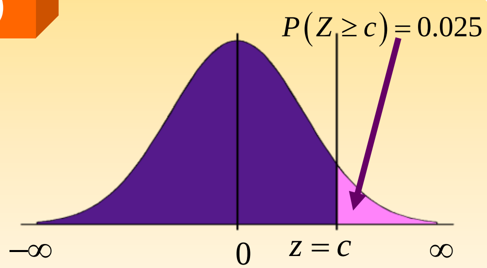
# EXAMPLE 5.7

If $Z$ is $N(0,1)$, find values of $c$ such that

(i) $P(Z \geq c) = 0.025.$ (ii) $P(|Z| \leq c) = 0.95.$ (iii) $P(|Z| \leq c) = 0.9011.$ (iv) $P(Z > c) = 0.0495.$

## SOLUTION

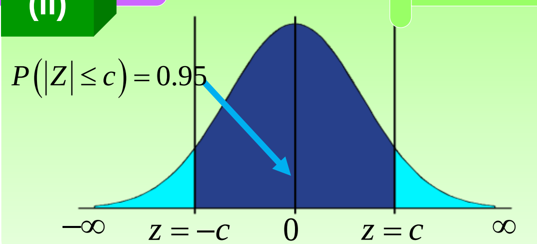
**(i)**




**Based on the Standard Normal Table**

$c = 1.9600$

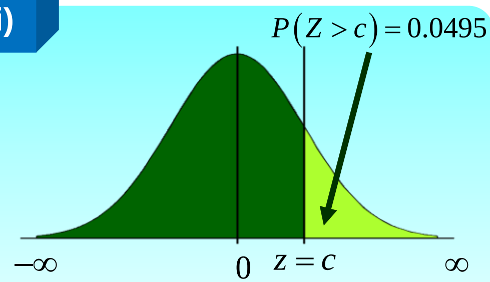
**(ii)**




**By on the Standard Normal Table**

$c = 1.9600$

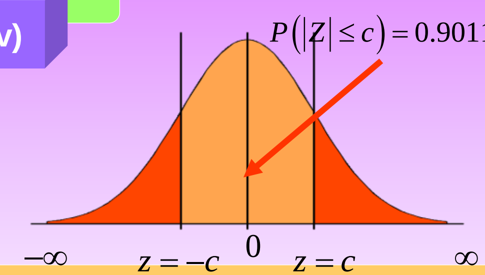
**(iii)**




**Based on the Standard Normal Table**

$c = 1.6500$

**(iv)**




**Based on the Standard Normal Table**

$c = 1.6500$

# EXERCISE 5.5

Given that the resistance of an electrical circuit follows the normal distribution with a mean of 10 ohms and standard deviation of 0.25 ohms. Denote that a *continuous random variable*, $X$ represents *the measurement of resistances in the terms of ohms*. Find

(i) $P(X < 9.5)$.  (ii) $P(9.6 < X < 10.4)$.

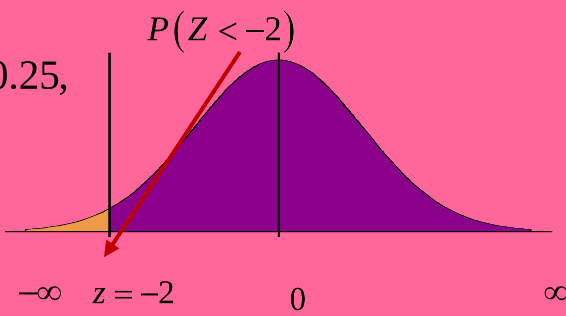
**SOLUTION**

**(i)**

Based on the question, we know that $\mu = 10$ and $\sigma = 0.25$,

$$P\left(\frac{X-\mu}{\sigma} < \frac{9.5-10}{0.25}\right) = P(Z < -2)$$

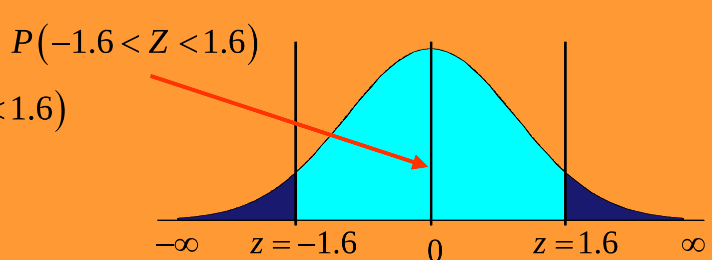
$$= 0.0228$$




**(ii)**

$$P\left(\frac{9.6-10}{0.25} < \frac{X-\mu}{\sigma} < \frac{10.4-10}{0.25}\right) = P(-1.6 < Z < 1.6)$$

$$= 0.8904$$



# EXERCISE 5.6

Denotes that the reduction of a person's oxygen consumption during sleep follows a **normal distribution** with a mean of 37.6 cc per minutes and standard deviation of 4.6 cc per minutes. **Find the probability** that the person's oxygen consumption during sleep will be reduced by

(i) at least 44.5 cc per minutes.
(ii) At most 35.0 cc per minutes.
(iii) anywhere from 30.0 to 40.0 cc per minutes.

**SOLUTION**

Based on the question, we know that

$$\mu = 37.6 \text{ cc per minute}; \ \sigma = 4.6 \text{ cc per minute}$$

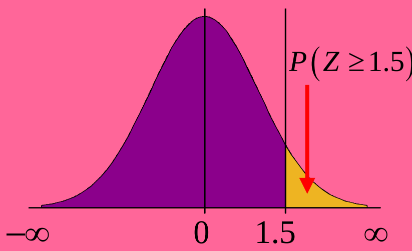
(i) $P(X \geq 44.5)$? (ii) $P(X \leq 35.0)$? (iii) $P(30.0 \leq X \leq 40.0)$?

**(i)**

**By using the function in calculator,**

$$P\left(\left(\frac{X-\mu}{\sigma}\right)\geq\left(\frac{44.5-37.6}{4.6}\right)\right)=P(Z\geq 1.5)$$

$$=0.0668$$

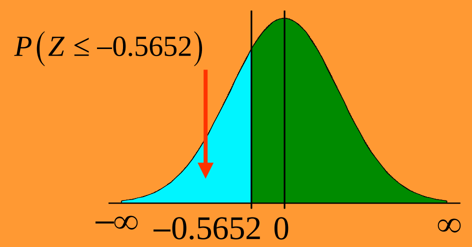



**(ii)**

**By using the function in calculator,**

$$P\left(\left(\frac{X-\mu}{\sigma}\right)\leq\left(\frac{35.0-37.6}{4.6}\right)\right)=P(Z\leq -0.5652)$$

$$=0.2860$$

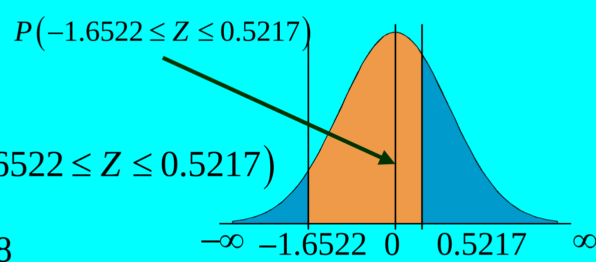



**(iii)**

**By using the function in calculator,**

$$P\left(\left(\frac{30.0-37.6}{4.6}\right)\leq\left(\frac{X-\mu}{\sigma}\right)\leq\left(\frac{40.0-37.6}{4.6}\right)\right)=P(-1.6522\leq Z\leq 0.5217)$$

$$=0.6498$$



# 5.4
# THE CENTRAL LIMIT THEOREM

# SAMPLING DISTRIBUTION

## CENTRAL LIMIT THEOREM (CLT)

If $\bar{X}$ is the mean of a random sample $X_1, X_2, \ldots, X_n$ of size $n$ from a distribution with finite mean $\mu$ and a positive finite variance $\sigma^2$, then the distribution of

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

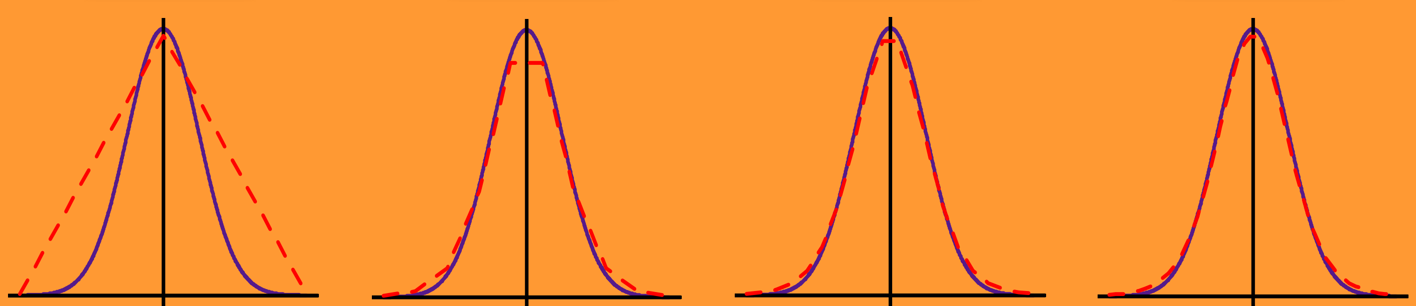
is $N(0,1)$ in the limit as $n \to \infty$.

**A ROUGH GUILDLINES:**
The sample size, $n \geq 30$ is consider large enough .

https://www.youtube.com/watch?v=Pujol1yC1_A

## SIMULATION

$n = 3$ | $n = 10$ | $n = 20$ | $n = 50$



# EXAMPLE 5.8

Given that a continuous random variable, $X$ represents the measurement of the diameter of the fetal head between 16th and 25th weeks of pregnancy is **normally distributed with a mean 46.58mm and standard deviation of 40.96mm.** Suppose that $\bar{X}$ represents *the sample mean of the random sample of size 16* for $X$.

(i) Determine the values of $\mathrm{E}(\bar{X})$ and $\mathrm{Var}(\bar{X})$.

(ii) Find $P(44.42 \le \bar{X} \le 48.98)$.

**SOLUTION**

**(i)**

$$\mathrm{E}(\bar{X}) = \mathrm{E}\left(\frac{1}{n}\sum_{i=1}^{16} X_i\right) = \frac{1}{n}\sum_{i=1}^{16}\mathrm{E}(X_i)$$

$$= \frac{1}{16}\left(16(46.58)\right)$$

$$= 46.58$$

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{16} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{16}\mathrm{Var}(X_i)$$

$$= \frac{1}{16^2}\left(16(40.96)\right)$$

$$= 2.56$$

**(ii)**

By using the function in calculator,

$$P(44.42 \le \bar{X} \le 48.98) = P\left(\frac{44.42-46.58}{\sqrt{2.56}} \le \frac{\bar{X}-\mu}{\sqrt{\sigma^2/n}} \le \frac{48.98-46.58}{\sqrt{2.56}}\right)$$

$$= P(-1.35 \le Z \le 1.50)$$

$$= 0.8447$$

# EXCERCISE 5.7

Suppose that the weight of the-born baby born in a community is adequately modeled by the normal distribution with $\mathrm{E}(X)=3320g$ and $\mathrm{Var}(X)=660^2$. Given that $\bar{X}$ the sample mean of a random sample of size 225. Based on the information, find $P(3233.76 \le \bar{X} \le 3406.24)$,

## SOLUTION

(i)

$$\mathrm{E}(\bar{X})=\mathrm{E}\left(\frac{1}{n}\sum_{i=1}^{225}X_i\right)=\frac{1}{n}\sum_{i=1}^{225}\mathrm{E}(X_i)$$
$$=\frac{1}{225}(225(3320))$$
$$=3320$$

$$\mathrm{Var}(\bar{X})=\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{225}X_i\right)=\frac{1}{n^2}\sum_{i=1}^{225}\mathrm{Var}(X_i)$$
$$=\frac{1}{225^2}(225(660^2))$$
$$=1936$$

(ii)

By using the function in calculator,

$$P(3233.76 \le \bar{X} \le 3406.24)=P\left(\frac{3233.76-3320}{\sqrt{1936}} \le \frac{\bar{X}-\mu}{\sqrt{\sigma^2/n}} \le \frac{3406.24-3320}{\sqrt{1936}}\right)$$
$$=P(-1.9600 \le Z \le 1.9600)$$
$$=0.9500$$

# EXERCISE 5.8

Given that a continuous random variable, $X$ represents the force required to push a crate across a factory, where $X \sim N\left(147.8,\ 12.3^2\right)$.

(i) Find $P\left(X < 163.3\right)$.

(ii) If $\bar{X}$ is the mean and $S^2$ is the variance of a 25 random sample from the distribution of $X$, find $P\left(\bar{X} \leq 150.9\right)$.

**By using the function in calculator,**

$$P\left(\left(\frac{X-\mu}{\sigma}\right)<\left(\frac{163.3-147.8}{12.3}\right)\right)=P\left(Z<1.2602\right)$$
$$=0.8962$$

(i)

$$\mathrm{E}\left(\bar{X}\right)=\mathrm{E}\left(\frac{1}{n}\sum_{i=1}^{25}X_i\right)=\frac{1}{n}\sum_{i=1}^{25}\mathrm{E}\left(X_i\right)$$
$$=\frac{1}{25}\left(25\left(147.8\right)\right)$$
$$=147.8$$

$$\mathrm{Var}\left(\bar{X}\right)=\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{25}X_i\right)=\frac{1}{n^2}\sum_{i=1}^{25}\mathrm{Var}\left(X_i\right)$$
$$=\frac{1}{25^2}\left(25\left(12.3^2\right)\right)$$
$$=6.0516$$

(ii)

**By using the function in calculator,**

$$P\left(\left(\frac{\bar{X}-\mu}{\sqrt{\sigma^2/n}}\right)\leq\left(\frac{150.9-147.8}{\sqrt{6.0516}}\right)\right)=P\left(Z\leq 1.2602\right)$$
$$=0.8962$$

(ii)

# 5.5
# NORMAL APPROXIMATION TO BINOMIAL DISTRIBUTION

# 5.6
# NORMAL APPROXIMATION TO POISSON DISTRIBUTION

# THE RELATIONSHIP BETWEEN THE NORMAL DISTRIBUTION AND THE BINOMIAL DISTRIBUTION

## THE NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

If $X$ is a random variable follows a binomial distribution with parameters, $n$ and $p$, there the normal distribution approximation to the binomial distribution

$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$

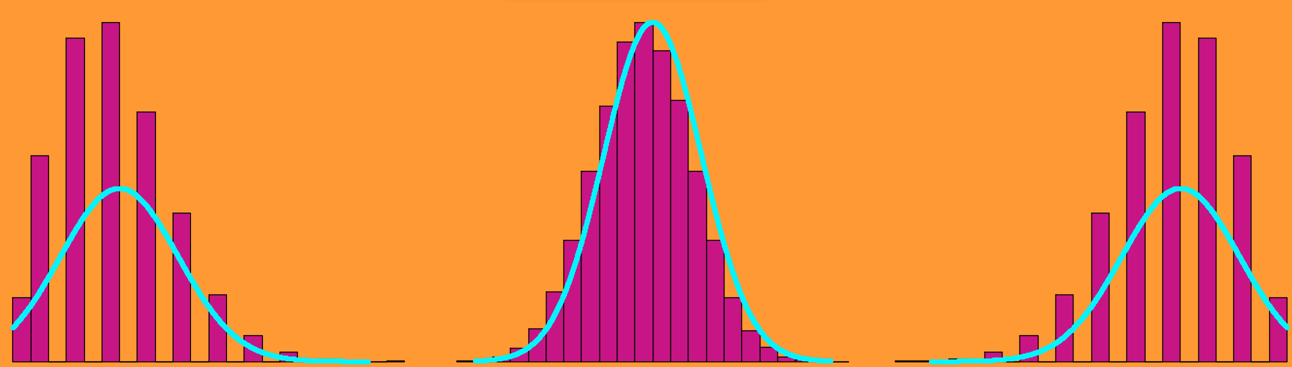
is $N(0,1)$ in the limit as $n \to \infty$.

**A ROUGH GUILDLINES:**
$np > 5$ and $n(1-p) > 5$

https://www.youtube.com/watch?v=CCqWkJ_pqNU&t=18s

## SIMULATION

$n = 30;\ p = 0.1$ $\quad$ $n = 30;\ p = 0.5$ $\quad$ $n = 30;\ p = 0.9$



# THE RELATIONSHIP BETWEEN A NORMAL DISTRIBUTION AND A POISSON DISTRIBUTION

## THE NORMAL APPROXIMATION TO THE POISSON DISTRIBUTION

If $X$ is a random variable follows a Poisson distribution with parameters, $\lambda$, therefore the normal approximation to the Poisson distribution is

$$Z = \frac{X - \lambda}{\sqrt{\lambda}}$$

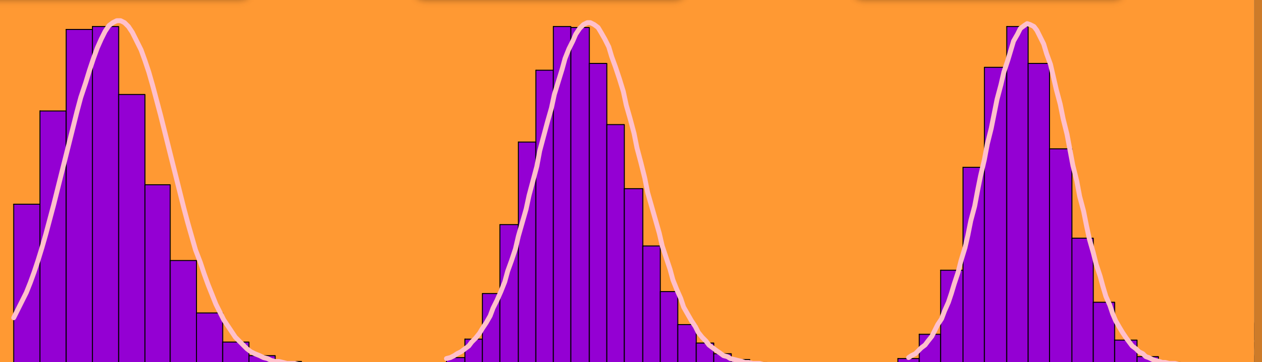
is $N(0,1)$ in the limit as $\lambda \to \infty$.

**A ROUGH GUILDLINES:**

$\lambda > 15$

## SIMULATION

$\lambda = 4$ $\lambda = 8$ $\lambda = 16$



# THE CONTINUITY CORRECTION FACTOR FOR DISCRETE DITRIBUTION

$P(X \le \alpha) \approx P(X \le \alpha + 0.5)$

$P(X > \alpha) \approx P(X > \alpha + 0.5)$

$P(X \ge \alpha) \approx P(X \ge \alpha - 0.5)$

$P(X < \alpha) \approx P(X < \alpha - 0.5)$

$P(X = \alpha) \approx P(\alpha - 0.5 < X < \alpha + 0.5)$

$P(\alpha \le X \le \beta) \approx P(\alpha + 0.5 \le X \le \beta + 0.5)$

$P(\alpha < X \le \beta) \approx P(\alpha + 0.5 < X \le \beta + 0.5)$

$P(\alpha < X < \beta) \approx P(\alpha - 0.5 < X < \beta - 0.5)$

$P(\alpha \le X < \beta) \approx P(\alpha - 0.5 \le X < \beta - 0.5)$

# EXAMPLE 5.9

Given that a ***discrete random variable***, $X$ **follows a binomial distribution** with parameters $n = 100$ and $p = 0.10.$ Find $P(12 \le X \le 14)$ by using

(i) the normal approximation to binomial distribution.

(ii) the normal approximation to Poisson distribution.

Based on probability mass function of binomial distribution ,

$$P(12 \le X \le 14) = P(X = 12) + P(X = 13) + P(X = 14) = 0.2244$$

More Accurate!

## SOLUTION

**Based on the question, we know that**

$$\mu \approx np = 100(0.1) = 10$$

$$\sigma^2 \approx npq = 100(0.1)(1-0.1) = 9$$

$$\therefore X \simeq N(x;10,9)$$

**Therefore,**

$$P(12 \le X \le 14) \approx P\left(\frac{12+0.5-10}{\sqrt{9}} \le \frac{X-\mu}{\sigma} \le \frac{14+0.5-10}{\sqrt{9}}\right)$$

$$\approx P(0.8333 \le Z \le 1.5000)$$

$$\approx 0.1355$$

**Based on the question, we know that**

$$\lambda \approx np = 100(0.1) = 10$$

$$\therefore X \simeq P(x;10)$$

**Therefore,**

$$P(12 \le X \le 14) \simeq P(X \le 14) - P(X \le 11)$$

$$\simeq 0.9165 - 0.6968$$

$$\simeq 0.2197$$

# EXAMPLE 5.10

A manufacturer of fabric found that ***the number of flaws of fabric in a production line*** is ***adequately modeled by a Poisson distribution*** with ***average of two***. For further quality improvement, the manufacturer has randomly selects ***50 of fabric from the production line***. By using a normal distribution approximation to Poisson distribution, ***determine the probability*** that ***the total number of flaws in these 50 fabric is less than 110***?

**SOLUTION**

**Based on the question, we know that**

$$\begin{aligned}\mu = \sigma^2 &\approx \lambda \\ &\approx 50(2) \\ &\approx 100\end{aligned}$$

**By using the function in calculator,**

$$\begin{aligned}P(X < 110) &\approx P\left(\frac{X-\mu}{\sigma} < \frac{110-0.5-100}{\sqrt{100}}\right) \\ &\approx P(Z < 0.95) \\ &\approx 0.8289\end{aligned}$$

# EXERCISE 5.9

In a community, it  found ***70% kids infected with food poisoning due to pathogens in unhygienic food***. Denotes that ***a continuous random variable***, $X$, which are ***normally distributed*** and given that in ***84 random samples of kids who infected by food poisoning***. ***By using a normal distribution approximation to binomial distribution***, find $P(X \le 52)$.

**SOLUTION**

**Based on question, we know that**

$$\mu \approx np = 84(0.70) = 58.8$$

$$\sigma^2 \approx np(1-p) = 17.64$$

**Therefore,**

$$X \simeq N(x; 58.8, 17.64)$$

**By using the function in calculator,**

$$P(X \le 52) \simeq P\left(\frac{X-\mu}{\sigma} \le \frac{52+0.5-58.8}{\sqrt{17.64}}\right)$$

$$\simeq P(Z \le -1.5000)$$

$$\simeq 0.0668$$

# THANK YOU

## END OF CHAPTER 5 (PART 2)