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DUM 2413 STATISTICS & PROBABILITY

CHAPTER 5
CONTINUOUS PROBABILITY DISTRIBUTIONS

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EXPECTED OUTCOMES

- Able to determine the expected value, standard deviation and variance of a continuous random variable
- Able to identify the relationship between the normal distribution and the sampling distribution of the mean
- Able to solve the application problems, which involved the normal distribution and the sampling distribution of the mean
- Able to identify the relationship between the Binomial and Poisson distribution with a standard normal distribution

CONTENT

5.1
CONTINUOUS RANDOM VARIABLE
AND
PROBABILITY DENSITY FUNCTION

PROBABILITY DENSITY FUNCTION

A function with values $f(x)$ defined over the set of all real numbers, is known as probability density function of the continuous random variable X defined as

$$P(\alpha \le X \le \beta) = \int_{\alpha}^{\beta} f(x)\,dx$$

for any real constants α and β with $\alpha \le \beta.$

If X is a continuous random variable and α and β are real constants with $\alpha \le \beta,$ then

$$P(\alpha \le X \le \beta) = P(\alpha \le X < \beta) = P(\alpha < X \le \beta) = P(\alpha < X < \beta)$$

A function can be defined as a probability density of a continuous random variable, X if its values, $f(x)$ satisfy the conditions

i. $f(x) \ge 0 \quad \text{for } -\infty < x < \infty.$

ii. $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

REMEMBER: $P(X = c) = 0$ for any real constant c.

EXAMPLE 5.1

A continuous random variable T has a probability density function g defined

$$g(t)=\begin{cases}\dfrac{3t^2}{35}, & -2\le t<3\\ 0, & \text{otherwise}\end{cases}$$

Show that T is a continuous random variable and g is a probability density function.

SOLUTION

$$\int_{-\infty}^{\infty} g(t)\,dt=\int_{-\infty}^{-2}(0)\,dt+\int_{-2}^{3}\left(\frac{3t^2}{35}\right)dt+\int_{3}^{\infty}(0)\,dt$$

$$=[0]_{-\infty}^{-2}+\left[\frac{t^3}{35}\right]_{-2}^{3}+[0]_{3}^{\infty}$$

$$=(0-0)+\left(\frac{27}{35}-\left(-\frac{8}{35}\right)\right)+(0-0)$$

$$=1$$

Therefore, T is a continuous random variable with g as a probability density function of T.

EXAMPLE 5.2

The continuous random variable Y has a probability density function f given by

$$f(y)=\begin{cases} y, & 0\le y<1 \\ 2-y, & 1\le y<2 \\ 0, & \text{otherwise} \end{cases}$$

Show that f is a probability density function.

SOLUTION

$$\int_{-\infty}^{\infty} f(y)\,dy=\int_{-\infty}^{0}(0)\,dy+\int_{0}^{1}(y)\,dy+\int_{1}^{2}(2-y)\,dy+\int_{2}^{\infty}(0)\,dy$$

$$=[0]_{-\infty}^{0}+\left[\frac{y^2}{2}\right]_0^1+\left[2y-\frac{y^2}{2}\right]_1^2+[0]_2^{\infty}$$

$$=(0-0)+\left(\frac{1^2}{2}-\frac{0^2}{2}\right)+\left(2(2)-\frac{2^2}{2}-\left(2(1)-\frac{1^2}{2}\right)\right)+(0-0)$$

$$=1$$

Therefore, Y is a continuous random variable with f as a probability density function of Y.

EXAMPLE 5.3

The continuous random variable X has a probability density function g given by

$$g(x)=\begin{cases} x^2, & 0<x<1 \\ \dfrac{x-2}{12}, & 2\le x<6 \\ 0, & \text{otherwise} \end{cases}$$

Show that g is a probability density function. Then, find

(i) $P(2\le X\le 4)$ (ii) $P\left(X<\frac{3}{4}\right)$ (iii) $P\left(\frac{1}{2}\le X<3\right)$

SOLUTION

$$\int_{-\infty}^{\infty} g(x)\,dx=\int_{-\infty}^{0}(0)\,dx+\int_{0}^{1}\left(x^2\right)dx+\int_{1}^{2}(0)\,dx+\int_{2}^{6}\left(\frac{x-2}{12}\right)dx+\int_{6}^{\infty}(0)\,dx$$

$$=[0]_{-\infty}^{0}+\left[\frac{x^3}{3}\right]_0^1+[0]_1^2+\left[\frac{x^2}{24}-\frac{x}{6}\right]_2^6+[0]_6^{\infty}$$

$$=(0-0)+\left(\frac{1^3}{3}-\frac{0^3}{3}\right)+(0-0)+\left(\frac{6^2}{24}-\frac{6}{6}-\left(\frac{2^2}{24}-\frac{2}{6}\right)\right)+(0-0)$$

$$=1$$

Therefore, X is a continuous random variable with g as a probability density function of X.

SOLUTION

(i)

$$P(2 \le X \le 4) = \int_2^4 \left(\frac{x-2}{12}\right) dx = \left[\frac{x^2}{2(12)} - \frac{2x}{12}\right]_2^4 = \left(\frac{4^2}{24} - \frac{4}{6}\right) - \left(\frac{2^2}{24} - \frac{2}{6}\right) = \frac{1}{6}$$

(ii)

$$P\left(X \le \frac{3}{4}\right) = \int_{-\infty}^{0} (0)\, dx + \int_0^{\frac{3}{4}} \left(x^2\right) dx = [0]_{-\infty}^{0} + \left[\frac{x^3}{3}\right]_0^{\frac{3}{4}} = (0-0) + \left(\frac{\left(\frac{3}{4}\right)^3}{3} - \frac{(0)^3}{3}\right) = \frac{9}{64}$$

(iii)

$$P\left(\frac{1}{2} \le X < 3\right) = \int_{\frac{1}{2}}^{1} \left(x^2\right) dx + \int_1^2 (0)\, dx + \int_2^3 \left(\frac{x-2}{12}\right) dx = \left[\frac{x^3}{3}\right]_{\frac{1}{2}}^{1} + [0]_1^2 + \left[\frac{x^2}{24} - \frac{x}{6}\right]_2^3$$

$$= \left(\frac{(1)^3}{3} - \frac{\left(\frac{1}{2}\right)^3}{3}\right) + (0-0) + \left(\frac{(3)^2}{24} - \frac{3}{6} - \left(\frac{(2)^2}{24} - \frac{2}{6}\right)\right) = \frac{1}{3}$$

EXERCISE 5.1

Suppose that g is a probability density function for a continuous random variable W, which defined as below.

$$g(w) = \begin{cases} \frac{1}{18}\left(w^2 - 2w\right), & 2 \le x < 5 \\ 0, & \text{otherwise} \end{cases}$$

Find

(i) $P(W \le 3)$

(ii) $P(|W| > 4)$

SOLUTION

(i)

$$P(W \le 3) = \frac{1}{18}\int_{-\infty}^{2}(0)\,dw + \frac{1}{18}\int_{2}^{3}\left(w^2 - 2w\right)dw$$

$$= \frac{1}{18}[0]_{-\infty}^{2} + \frac{1}{18}\left[\frac{w^3}{3} - \frac{2w^2}{2}\right]_2^3$$

$$= (0-0) + \frac{1}{18}\left(\frac{3^3}{3} - 3^2 - \left(\frac{2^3}{3} - 2^2\right)\right)$$

$$= \frac{2}{27}$$

(ii)

$$P(|W| > 4) = P(W < -4) + P(W > 4) = \frac{1}{18}\int_{-\infty}^{-4}(0)\,dw + \frac{1}{18}\int_{4}^{5}\left(w^2 - 2w\right)dw + \frac{1}{18}\int_{5}^{\infty}(0)\,dw$$

$$= \frac{1}{18}[0]_{-\infty}^{-4} + \frac{1}{18}\left[\frac{w^3}{3} - \frac{2w^2}{2}\right]_4^5 + \frac{1}{18}[0]_5^{\infty}$$

$$= (0-0) + \frac{1}{18}\left(\frac{5^3}{3} - 5^2 - \left(\frac{4^3}{3} - 4^2\right)\right) + (0-0)$$

$$= \frac{17}{27}$$

EXERCISE 5.2

Let f is a probability density function for a continuous random variable W, which defined as follows.

$$f(w) = \begin{cases} \dfrac{10-w}{42}, & 0 < w \leq 6 \\ 0, & \text{otherwise} \end{cases}$$

(i) Given that $P(W > u) = \dfrac{9}{84}$, find u.

(ii) Find $P(2 < W < 4)$.

SOLUTION

(i)

$$P(W>u)=\int_u^6\left(\frac{10-w}{42}\right)dw+\int_6^\infty(0)\,dw=\frac{9}{84}$$

$$\frac{1}{42}\left[10w-\frac{w^2}{2}\right]_u^6+[0]_6^\infty=\frac{9}{84}$$

$$\left(60-\frac{36}{2}\right)-\left(10u-\frac{u^2}{2}\right)=\frac{9}{2}$$

$$u^2-20u+75=0$$

$$u^2-20u+75=0$$

$$(u-15)(u-5)=0$$

$$u=15,\ u=5$$

Since $0<w\le 6$, therefore $u=15$ is invalid.

$\therefore u=5$.

(ii)

$$P(2<W<4)=\int_2^4\left(\frac{10-w}{42}\right)dw$$

$$=\left[\frac{10w}{42}-\frac{w^2}{84}\right]_2^4$$

$$=\left(\frac{10(4)}{42}-\frac{4^2}{84}\right)+\left(\frac{10(2)}{42}-\frac{2^2}{84}\right)$$

$$=\frac{1}{3}$$

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5.2
MEAN AND VARIANCE

MEAN

If X is a **continuous random variable** and $f(x)$ is the its **probability density** at x, therefore, the expected value of X is defined as

$$\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} \left(x \cdot f(x)\right) dx$$

VARIANCE

The variance of a **continuous random variable** X is defined as

$$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \left(E(X)\right)^2$$

$$= \int_{-\infty}^{\infty} \left(x^2 \cdot f(x)\right) dx - \mu^2$$

SOME PROPERTIES OF MEAN AND VARIANCE

Let α and β be constants. Then the following results holds:

(i) $\mathrm{E}(\alpha) = \alpha$ **(ii)** $\mathrm{E}(\alpha X) = \alpha\,\mathrm{E}(X)$

(iii) $\mathrm{Var}(\alpha X + \beta) = \alpha^2\,\mathrm{Var}(X)$. **In particular,** $\mathrm{Var}(\alpha X) = \alpha^2\,\mathrm{Var}(X)$, $\mathrm{Var}(\beta) = 0$.

EXAMPLE 5.4

Given the probability density function of a continuous random variable, X is

$$f(x)=\begin{cases}2x, & 0\le x\le 1\\ 0, & \text{otherwise}\end{cases}$$

Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. Hence, find

(i) $\mathrm{E}(2X+5)$ (ii) $\mathrm{E}(X+4)$ (iii) $\mathrm{Var}(X+3)$ (iv) $\mathrm{Var}(3X-2)$

SOLUTION

$$\begin{aligned}\mu=\mathrm{E}(X)&=\int_{-\infty}^{\infty}\left(x\cdot f(x)\right)\,dx\\&=\int_{-\infty}^{0}(x\cdot 0)\,dx+\int_{0}^{1}(x\cdot 2x)\,dx+\int_{1}^{\infty}(x\cdot 0)\,dx\\&=\int_{-\infty}^{0}(0)\,dx+\int_{0}^{1}\left(2x^2\right)dx+\int_{1}^{\infty}(0)\,dx\\&=[0]_{-\infty}^{0}+\left[\frac{2x^3}{3}\right]_0^1+[0]_1^{\infty}\\&=(0-0)+\left(\frac{2(1)^3}{3}-\frac{2(0)^3}{3}\right)+(0-0)\\&=\frac{2}{3}\end{aligned}$$

$$\begin{aligned}\mathrm{E}\left(X^2\right)&=\int_{-\infty}^{\infty}\left(x^2\cdot f(x)\right)dx\\&=\int_{-\infty}^{0}\left(x^2\cdot 0\right)dx+\int_{0}^{1}\left(x^2\cdot 2x\right)dx+\int_{1}^{\infty}\left(x^2\cdot 0\right)dx\\&=\int_{-\infty}^{0}(0)\,dx+\int_{0}^{1}\left(2x^3\right)dx+\int_{1}^{\infty}(0)\,dx\\&=[0]_{-\infty}^{0}+\left[\frac{2x^4}{4}\right]_0^1+[0]_1^{\infty}\\&=(0-0)+\left(\frac{2(1)^4}{4}-\frac{2(0)^4}{4}\right)+(0-0)\\&=\frac{1}{2}\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(X)&=\mathrm{E}\left(X^2\right)-\left[\mathrm{E}(X)\right]^2\\&=\frac{1}{2}-\left(\frac{2}{3}\right)^2\\&=\frac{1}{18}\end{aligned}$$

(i)

$$\mathrm{E}(2X+5)=2\,\mathrm{E}(X)+E(5)=2\left(\frac{2}{3}\right)+5=\frac{19}{3}$$

(ii)

$$\mathrm{E}(X+4)=\mathrm{E}(X)+\mathrm{E}(4)=\frac{2}{3}+4=\frac{14}{3}$$

(iii)

$$\mathrm{Var}(X+3)=\mathrm{Var}(X)+\mathrm{Var}(3)=\frac{1}{18}+0=\frac{1}{18}$$

(iv)

$$\mathrm{Var}(3X-2)=\mathrm{Var}(3X)+\mathrm{Var}(2)=3^2\,\mathrm{Var}(X)+0=9\left(\frac{1}{18}\right)=\frac{1}{2}$$

EXAMPLE 5.5

Y is the continuous random variable that has a probability density function as follows

$$g(y)=\begin{cases}\dfrac{2y}{3}, & 0\le y<1\\ \dfrac{3-y}{3}, & 1\le y<3\\ 0, & \text{otherwise}\end{cases}$$

Find

(i) the expected value.

(ii) the variance and the standard deviation.

SOLUTION

Mean

$$\mu=\mathrm{E}(Y)=\int_{-\infty}^{\infty}\left(y\cdot g(y)\right)\,dy=\int_{-\infty}^{0}(y\cdot 0)\,dy+\int_{0}^{1}\left(y\cdot\frac{2y}{3}\right)dy+\int_{1}^{3}\left(y\cdot\frac{3-y}{3}\right)dy+\int_{3}^{\infty}(y\cdot 0)\,dy$$

$$=[0]_{-\infty}^{0}+\left[\frac{2y^3}{3(3)}\right]_0^1+\left[\frac{3y^2}{3(2)}-\frac{y^3}{3(3)}\right]_1^3+[0]_3^{\infty}$$

$$=(0-0)-\left(\frac{2(1)^3}{9}-\frac{2(0)^3}{9}\right)+\left(\frac{3^2}{2}-\frac{3^3}{9}-\left(\frac{1^2}{2}-\frac{1^3}{9}\right)\right)+(0-0)=\frac{4}{3}$$

(i)

$$\mu = \mathrm{E}\left(Y^2\right) = \int_{-\infty}^{\infty}\left(y^2 \cdot g\left(y\right)\right)\,dy = \int_{-\infty}^{0}\left(y^2 \cdot 0\right)\,dy + \int_{0}^{1}\left(y^2 \cdot \frac{2y}{3}\right)\,dy + \int_{1}^{3}\left(y^2 \cdot \frac{3-y}{3}\right)\,dy + \int_{3}^{\infty}\left(y^2 \cdot 0\right)\,dy$$

$$= [0]_{-\infty}^{0} + \left[\frac{2y^4}{3(4)}\right]_0^1 + \left[\frac{3y^3}{3(3)} - \frac{y^4}{3(4)}\right]_1^3 + [0]_3^{\infty}$$

$$= (0-0) - \left(\frac{(1)^4}{6} - \frac{(0)^4}{6}\right) + \left(\frac{3^3}{3} - \frac{3^4}{12} - \left(\frac{1^3}{3} - \frac{1^4}{12}\right)\right) + (0-0) = \frac{13}{6}$$

(i)

Variance

$$\sigma^2 = \mathrm{Var}\left(Y\right) = \mathrm{E}\left(Y^2\right) - \left[\mathrm{E}\left(Y\right)\right]^2$$

$$= \frac{13}{6} - \left(\frac{4}{3}\right)^2$$

$$= \frac{7}{18}$$

Standard Deviation

$$\sigma = \sqrt{\mathrm{Var}\left(Y\right)}$$

$$= \sqrt{\frac{7}{18}}$$

$$= 0.6236$$

(ii)

EXERCISE 5.3

Given the probability density function of a *continuous random variable*, X is

$$f(x)=\begin{cases} -\alpha\left(x^2-4\right), & 0\le x<2 \\ 0, & \text{otherwise} \end{cases}$$

where α is a constant.

(i) Show that $\alpha=\dfrac{3}{16}$.

(ii) Find the expected value and variance of X.

SOLUTION

$$\int_{-\infty}^{\infty} f(x)\,dx=\int_{-\infty}^{0}(0)\,dx+\int_{0}^{2}\left(-a\left(x^2-4\right)\right)dx+\int_{2}^{\infty}(0)\,dx=[0]_{-\infty}^{0}+\left[-\frac{ax^3}{3}+4ax\right]_0^2+[0]_2^{\infty}=1$$

$$(0-0)+\left(-\frac{a(2)^3}{3}+4(2)a-\left(-\frac{a(0)^3}{3}+4(0)a\right)\right)+(0-0)=1$$

$$\frac{16a}{3}a=1$$

$$a=\frac{3}{16}$$

(i)

Mean

$$\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} \left(x \cdot f(x)\right)\, dx = \int_{-\infty}^{2} (x \cdot 0)\, dx + \int_{0}^{2} \left(x \cdot \left(\frac{3}{4} - \frac{3}{16}x^2\right)\right) dx + \int_{2}^{\infty} (x \cdot 0)\, dx$$

$$= [0]_{-\infty}^{0} + \left[\frac{3x^2}{4(2)} - \frac{3x^4}{16(4)}\right]_0^2 + [0]_2^{\infty}$$

$$= (0-0) + \left(\frac{3(2)^2}{8} - \frac{3(2)^4}{64} - \left(\frac{3(0)^2}{8} - \frac{3(0)^4}{64}\right)\right) + (0-0) = \frac{3}{4}$$

(ii)

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} \left(x^2 \cdot f(x)\right)\, dx = \int_{-\infty}^{2} (x^2 \cdot 0)\, dx + \int_{0}^{2} \left(x^2 \cdot \left(\frac{3}{4} - \frac{3}{16}x^2\right)\right) dx + \int_{2}^{\infty} (x^2 \cdot 0)\, dx$$

$$= [0]_{-\infty}^{0} + \left[\frac{3x^3}{4(3)} - \frac{3x^5}{16(5)}\right]_0^2 + [0]_2^{\infty}$$

$$= (0-0) + \left(\frac{(2)^3}{4} - \frac{3(2)^5}{80} - \left(\frac{(0)^3}{4} - \frac{3(0)^5}{80}\right)\right) + (0-0) = \frac{4}{5}$$

(ii)

Variance

$$\sigma^2 = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

$$= \frac{4}{5} - \left(\frac{3}{4}\right)^2$$

$$= \frac{19}{80}$$

(ii)

EXERCISE 5.4

Given the probability density function of a continuous random variable T is

$$g(t)=\begin{cases}\gamma\left(4t-t^2+12\right), & 0\le t<6\\ 0, & \text{otherwise}\end{cases}$$

where γ is a constant. Find

(i) the value of γ.

(ii) $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

SOLUTION

$$\int_{-\infty}^{\infty} g(t)\,dt=\int_{-\infty}^{0}(0)\,dt+\int_{0}^{6}\left(4tc-t^2c+12c\right)dt+\int_{6}^{\infty}(0)\,dt=[0]_{-\infty}^{0}+\left[\frac{4ct^2}{2}-\frac{ct^3}{3}+12ct\right]_0^6+[0]_6^{\infty}=1$$

$$(0-0)+\left(\left(2(6)^2c-\frac{(6)^3}{3}c+12(6)c-\left(2(0)^2c-\frac{(0)^3}{3}c+12(0)c\right)\right)\right)+(0-0)=1$$

$$72c=1$$

$$c=\frac{1}{72}$$

(i)

Mean

$$\mu = \mathrm{E}(T) = \int_{-\infty}^{\infty} \left(t \cdot g(t)\right)\, dt = \int_{-\infty}^{0} (t \cdot 0)\, dt + \frac{1}{72}\int_{0}^{6} \left(t \cdot \left(4t - t^2 + 12\right)\right) dt + \int_{6}^{\infty} (t \cdot 0)\, dt$$

$$= [0]_{-\infty}^{0} + \frac{1}{72}\left[\frac{4t^3}{3} - \frac{t^4}{4} + \frac{12t^2}{2}\right]_0^6 + [0]_6^{\infty}$$

$$= (0-0) + \frac{1}{72}\left(\frac{4(6)^3}{3} - \frac{(6)^4}{4} + 6(6)^2 - \left(\frac{4(0)^3}{3} - \frac{(0)^4}{4} + 6(0)^2\right)\right) + (0-0) = \frac{5}{2}$$

(ii)

$$\mathrm{E}\left(T^2\right) = \int_{-\infty}^{\infty} \left(t^2 \cdot g(t)\right) dt = \int_{-\infty}^{0} \left(t^2 \cdot 0\right) dt + \frac{1}{72}\int_{0}^{6} \left(t^2 \cdot \left(4t - t^2 + 12\right)\right) dt + \int_{6}^{\infty} \left(t^2 \cdot 0\right) dt$$

$$= [0]_{-\infty}^{0} + \frac{1}{72}\left[\frac{4t^4}{4} - \frac{t^5}{5} + \frac{12t^3}{3}\right]_0^6 + [0]_6^{\infty}$$

$$= (0-0) + \frac{1}{72}\left((6)^4 - \frac{(6)^5}{5} + 4(6)^3 - \left((0)^4 - \frac{(0)^5}{5} + 4(0)^3\right)\right) + (0-0) = \frac{42}{5}$$

(ii)

Variance

$$\sigma^2 = \mathrm{E}\left(T^2\right) - \left[\mathrm{E}(T)\right]^2$$

$$= \frac{42}{5} - \left(\frac{5}{2}\right)^2$$

$$= \frac{43}{20}$$

(ii)

THANK YOU

END OF CHAPTER 5 (PART 1)