DUM 2413 STATISTICS & PROBABILITY

CHAPTER 4

DISCRETE PROBABILITY DISTRIBUTIONS

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EXPECTED OUTCOMES

- Able to determine the expected value, standard deviation and variance of a discrete random variable
- Able to distinguish between the binomial and Poisson distributions
- Able to solve the application problems, which involved the binomial and Poisson distributions
- Able to identify the relationship between the binomial and Poisson distributions

CONTENT

4.3
BINOMIAL DISTRIBUTION

4.4
POISSON DISTRIBUTION

DISCRETE PROBABILITY DISTRIBUTIONS

DISCRETE PROBABILITY DISTRIBUTIONS

TYPE

BINOMIAL DISTRIBUTION

POISSON DISTRIBUTION

WHAT IS THE RELATIONSHIP BETWEEN
THE BINOMIAL DISTRIBUTION AND THE POISSON
DISTRIBUTION?

BINOMIAL DISTRIBUTION

PROBABILITY DISTRIBUTIONS

A *discrete random variable,* X is said to follow *binomial probability distribution* with parameters n and p, denoted by $bin(x;n,p)$ if

$$f(x) = P(X = x) = \binom{n}{x}(p)^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

where p is *probability of success* and $q = 1 - p$ is *probability of failure* on *a single trial*.

MEAN AND VARIANCE

If X is *binomial random variable* with parameters n and p, then

Mean, $\mathrm{E}(X) = \mu = np$

Variance, $\mathrm{Var}(X) = \sigma^2 = np(1-p)$

EXAMPLE 4.8

The previous records of an insurance company show that there are *20% of residents in a community do not have health insurance*. Suppose that *a random variable, X* represents *the number of residents do not have health insurance* in a random sample *15 residents*.

(i) How is *X* distributed?

(ii) Determine the *mean*, *variance* and *standard deviation* of *X*.

(iii) Find $P(X \geq 5)$.

SOLUTION

i

Since

$$n = 15$$

$$p = \frac{20}{100} = 0.20$$

Therefore, the distribution of *X*

$$X \sim bin(x; n, p)$$

$$X \sim bin(x; 15, 0.20)$$

ii

Mean :

$$\mu = \mathrm{E}(X) = np$$
$$= 15(0.20)$$
$$= 3.00$$

Variance :

$$\sigma^2 = \mathrm{Var}(X) = np(1-p)$$
$$= 15(0.20)(1-0.20)$$
$$= 2.40$$

Standard Deviation :

$$\sigma = \sqrt{np(1-p)}$$
$$= \sqrt{15(0.20)(0.80)}$$
$$= 1.5492$$

iii

$$P(X \geq 5) = 1 - P(X \leq 4)$$
$$= 1 - \sum_{x=0}^{4} \binom{15}{x} (0.2)^x (1-0.2)^{(15-x)}$$
$$= 1 - (0.0352 + 0.1319 + 0.2309 + 0.2501 + 0.1876)$$
$$= 0.1643$$

EXERCISE 4.6

In a research conducted about the satisfaction of among students to their lecturers' teaching skills. Based on analysis results, it was found that the *proportion of all students who satisfied with the lecturers' teaching skills is* $p = 0.10.$ Given that a *random variable, X* represents *the number of students out of a random sample of size* ***nine who do not satisfy the lecturers' teaching skills.***

(i) How is *X* distributed?

(ii) Determine the values of the *mean*, *variance* and *standard deviation* of *X*.

(iii) Find $P(X = 2)$ and $P(X \geq 2).$

SOLUTION

i

Since

$n = 9$

$p = 0.10$

Therefore, the distribution of *X*

$X \sim bin(x; n, p)$

$X \sim bin(x; 9, 0.10)$

ii

Mean:

$$\mu = \mathrm{E}(X) = np = 9(0.10) = 0.9000$$

Variance:

$$\sigma^2 = \mathrm{Var}(X) = np(1-p) = 9(0.10)(1-0.10) = 0.8100$$

Standard Deviation:

$$\sigma = \sqrt{np(1-p)} = \sqrt{9(0.10)(0.90)} = 0.9000$$

iii

$$P(X = x) = \binom{n}{x}(p)^x(1-p)^{(n-x)}$$

$$P(X = 2) = \binom{9}{2}(0.10)^2(1-0.10)^{(9-2)} = 0.1722$$

$$P(X \geq 2) = 1 - P(X \leq 1) = 1 - \sum_{x=0}^{1}\binom{9}{x}(0.10)^x(1-0.10)^{(9-x)} = 1 - 0.7748 = 0.2252$$

EXERCISE 4.7

During a manufacturing process in particular production line, the ***probability to produce a defective computer chip is 0.05***. For continuous quality improving, a quality engineer ***randomly selects six computer chip*** for the production line. Suppose that ***a random variable, X*** represents ***the number of defective computer chips in the sample..***

(i) How ***X*** is distributed?

(ii) Determine the values of ***E(X)*** and ***Var (X)***.

(iii) Find $P(X=0)$, and $P(X \geq 2)$, respectively.

SOLUTION

i

Since

$n = 6$

$p = 0.05$

Therefore, the distribution of *X*

$X \sim bin(x; n, p)$

$X \sim bin(x; 6, 0.05)$

ii

Mean :

$$\begin{aligned}\mu = \mathrm{E}(X) &= np \\ &= 6(0.05) \\ &= 0.3000\end{aligned}$$

Variance :

$$\begin{aligned}\sigma^2 = \mathrm{Var}(X) &= np(1-p) \\ &= 6(0.05)(1-0.05) \\ &= 0.2850\end{aligned}$$

iii

$$P(X = x) = \binom{n}{x}(p)^x(1-p)^{n-x}$$

$$\begin{aligned}P(X = 0) &= \binom{6}{0}(0.05)^0(1-0.05)^6 \\ &= 0.7351\end{aligned}$$

iii

$$P(X \geq 2) = 1 - P(X \leq 1) = 1 - \sum_{x=0}^{1}\binom{6}{x}(0.05)^x(1-0.05)^{(6-x)}$$

$$= 0.0328$$

EXERCISE 4.8

A *random variable, X* has a *binomial distribution* with *mean 6* and *variance 3.6*. $P(X=4)$.

SOLUTION

$$\mu = np = 6$$

$$\sigma^2 = npq = 3.6$$

$$\frac{\sigma^2}{\mu} = \frac{npq}{np} \Rightarrow q = \frac{3.6}{6} = 0.6000$$

$$p = 1 - q = 0.4000$$

$$\mu = np \Rightarrow n = \frac{6}{0.4000} = 15$$

$$P(X=4) = \binom{15}{4}(0.4000)^4(1-0.4000)^{(15-4)} = 0.1268$$

EXERCISE 4.9

The standard average weight of a grade A egg is 67.5 grams. Given that the probability of the egg weight less than 67.5 grams is 0.90. and let a random variable, X represents the number of eggs, which the weights less than 67.5 grams in a random sample of 8 eggs. Find

(i) $P(X=8)$.

(ii) $P(X\le 6)$.

(iii) $P(X\ge 6)$.

SOLUTION

Based on the question, we know that

$$n=8;\ p=0.90 \Rightarrow X \sim bin(x;8,0.90)$$

POISSON DISTRIBUTION

PROBABILITY DISTRIBUTIONS

A ***discrete random variable***, X is said to follow ***Poisson probability distribution*** with parameters, $\lambda > 0$, denoted by $Po(x;\ \lambda)$ if

$$f(x) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0,\ 1,\ 2,\ \ldots$$

MEAN AND VARIANCE

If X is a ***Poisson random variable*** with parameters λ, then

Mean, $\mathrm{E}(X) = \mu = \lambda$

Variance, $\mathrm{Var}(X) = \sigma^2 = \lambda$

EXAMPLE 4.9

Denotes that *a random variable, X* is *Poisson distributed* with *a mean of 4*. Find

(i) $P(2 \le X \le 5)$ (ii) $P(X \ge 3)$ (iii) $P(X \le 3)$

SOLUTION

i

$$P(2 \le X \le 5) = \sum_{x=2}^{5} \frac{e^{-4}(4)^x}{X!} = 0.1465 + 0.1954 + 0.1954 + 0.1563$$
$$= 0.6936$$

ii

$$P(X \ge 3) = 1 - P(X \le 2) = 1 - \sum_{x=0}^{2} \frac{e^{-4}4^x}{x!}$$
$$= 0.7619$$

iii

$$P(X \le 3) = \sum_{x=0}^{3} \frac{e^{-4}4^x}{x!} = 0.4335$$

EXAMPLE 4.10

The mean rate of a customer arrive at a bank is *11 per hour*. Support that that *the number of arrivals per hour* follows a *Poisson distribution*. Based on the given information, *find* the *probability that more than 10 customers arrive in a given hour*.

SOLUTION

Based on the question, we know that $\lambda = 11,$

$$P(X > 10) = 1 - P(X \leq 10) = 1 - \sum_{x=0}^{10} \frac{e^{-11} 11^x}{x!}$$

$$= 0.5401$$

EXAMPLE 4.11

BEWARE

In a manufacturing process, the *defective rate* of sheet metal is *five defects per 10 square feet*, which follows a Poisson distribution. What is *the probability that a 15-square-foot sheet of the metal* will have *at least six defects*?

SOLUTION

Based on the question, we know that

$$\lambda = \left(\frac{15}{10}\right)(5) = 7.5$$

$$P(X \geq 6) = 1 - P(X \leq 5) = 1 - \sum_{x=0}^{5} \frac{e^{-7.5} 7.5^{x}}{x!}$$

$$= 0.7585$$

EXERCISE 4.10

The ***average number of kangaroo ships*** **arriving on any one day at a jetty is known to be** ***12***. ***Find the probability that on a given day fewer than nine trucks will arrive at these jetty*****?**

SOLUTION

Based on the question, we know that

$$\lambda = 12$$

$$P(X < 9) = P(X \le 8) = \sum_{x=0}^{8} \frac{e^{-12} 12^x}{x!}$$

$$= 0.1550$$

EXERCISE 4.11

In a community, *the number of persons infected by stomach cancer for each year* ***follows a Poisson distribution*** with ***average 5.2.*** Find the probabilities of

(i) three stomach cancer infection in a given year.

(ii) at least 10 stomach infection in a given year.

(iii) anywhere from four to six stomach infection in a given year.

SOLUTION

i

Based on the question, we know that $\lambda = 5.2,$

$$P(X=3)=\frac{\left(e^{-\lambda}\right)\left(\lambda^{x}\right)}{x!}=\frac{\left(e^{-5.2}\right)\left(5.2^{3}\right)}{3!}=0.1293$$

ii

$$P(X \geq 10)=1-P(X \leq 9)=1-\sum_{x=0}^{9} \frac{e^{-5.2} 5.2^{x}}{x!}$$

$$=0.0396$$

iii

$$P(4 \leq X \leq 6)=\sum_{x=4}^{6} \frac{e^{-5.2} 5.2^{x}}{x!}=0.4944$$

EXERCISE 4.12

A 2 feet wide flat aluminum screen has an *average one flaw in a 100-foot roll*. Find the ***probability*** that a *50-foot roll has no flaws*.

BEWARE

SOLUTION

Based on the question, we know that

$$\lambda = \left(\frac{50}{100}\right)(1) = 0.5$$

Therefore,

$$P(X=0) = \frac{\left(e^{-\lambda}\right)\left(\lambda^{x}\right)}{x!} = \frac{\left(e^{-0.5}\right)\left(0.5^{0}\right)}{0!} = 0.6065$$

THE RELATIONSHIP BETWEEN THE BINOMIAL DISTRIBUTION AND THE POISSON DISTRIBUTION

THE POISSION DISTRIBUTION APPROXIMATION TO THE BINOMIAL DISTRIBUTION

THEOREM

If X is a ***binomial random variable*** with parameters n and p, then for each value $x = 0,\ 1,\ 2,\ \ldots$ and as $p \to 0,\ n \to \infty$ with $np \approx \lambda$ constant,

$$\lim_{x\to\infty}\binom{n}{x}(p)^x(1-p)^{n-x} \approx \frac{e^{-(np)}(np)^x}{x!}$$

ROUGH GUIDELINE

In some reference books, a ***rough guideline*** is **PROVIDED**, namely the ***Poisson distribution approximation*** to the ***Binomial distribution*** should be ***satisfied the condition***

i. $n > 50$ and $p < 0.1$

ii. $np < 5$

REMEMBER: This is just a rough guideline, therefore some of the reference books might be provided varies values of n and p.

EXAMPLE 4.12

The previous records show that the percentage of residents in a community infected by a rare disease *probability is 0.005 percent.* By using *the Poisson distribution approximation to the binomial distribution,* find the probability that among *10,000 residents in the community*,

(i) exactly two residents will be infected by the rare disease.

(ii) at most two residents will be infected by the rare disease.

SOLUTION

Based on the question, we know that

$$\lambda \approx np \approx 10000\left(\frac{0.005}{100}\right) \approx 0.5$$

i

$$P(X=2) \approx \frac{\left(e^{-0.5}\right)\left(0.5^{2}\right)}{2!} \approx 0.0758$$

ii

$$P(X \le 2) \approx \sum_{x=0}^{2} \frac{e^{-0.5}0.5^{x}}{x!} \approx 0.9856$$

EXERCISE 4.13

The percentage of all new licensed drivers involved in at one accident is *3 percent for any given year*. By using *the Poisson approximation to the binomial distribution*, find *the probability* that *among 150 new licensed drivers,*

(i) only five new licensed drivers involved in at least one accident for any given year.

(ii) at most three new licensed drivers involved in at least one accident for any given year.

SOLUTION

Based on the question, we know that

$$\lambda \approx np \approx 150\left(\frac{3}{100}\right) \approx 4.5$$

i

$$P(X=5)=\frac{\left(e^{-\lambda}\right)\left(\lambda^{x}\right)}{x!}=\frac{\left(e^{-4.5}\right)\left(4.5^{5}\right)}{5!}=0.1708$$

ii

$$P(X\le 3)=\sum_{x=0}^{3}\frac{e^{-4.5}4.5^{x}}{x!}=0.3423$$

THANK YOU

END OF CHAPTER 4 (PART 2)