Calculus
Applications of Integration

By

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Description

Aims

This chapter is aimed to :

1. Consider various application of integration
2. evaluate the definite and indefinite integral
3. explain the basic properties of integral

Expected Outcomes

1. Students should be able to use definite integral to find area between two curves
2. Students should be able to sketch graph to find area between curve and surface area
3. Students should be able to determine the length of a plane curve

References

1. Abdul Wahid Md Raji, Hamisan Rahmat, Ismail Kamis, Mohd Nor Mohamad, Ong Chee Tiong*. **The First Course of Calculus for Science & Engineering Students***, Second Edition, UTM 2016.

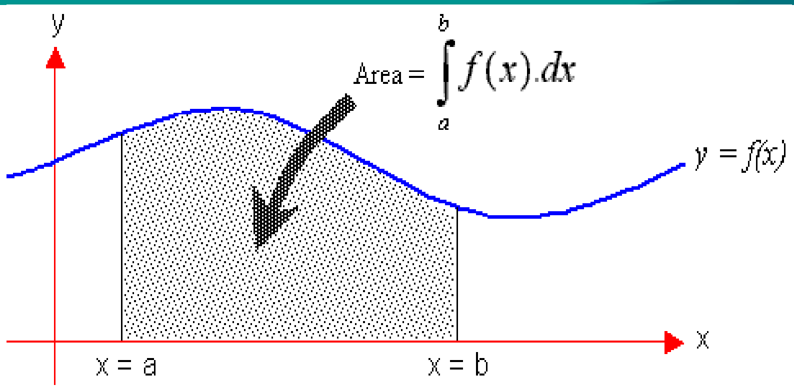
Content

1 Area Under a Curve

2 Area Between a Curves

3 Arc Length and Surface Area

Area Under the Curve

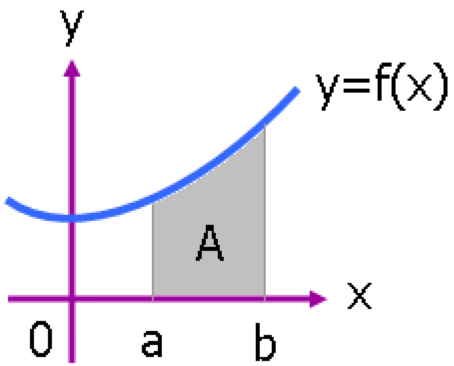


Definition – Area: If $f(x)$ continuous on $[a,b]$ then the area of region between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is given by

$$A = \int_a^b f(x)dx$$

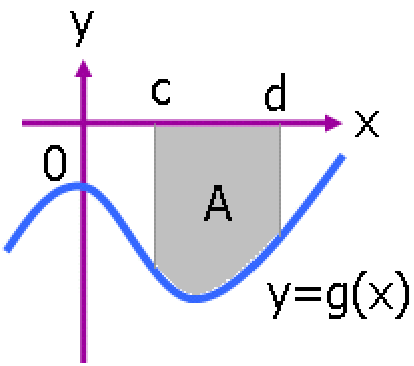
.

Above $x - \text{axis}$



$$\text{Area, A} = \int_a^b f(x)\,dx$$

Below $x - \text{axis}$



$$\text{Area, A} = \left| \int_c^d g(x)\,dx \right|$$

Example

Find the area bounded by the curve $y = x^2$ and x-axis between $x = 0$ to $x = 3$.

The required region is the shaded region as shown in Figure 1. The area of this region is

$$\int_0^3 x^2 dx = \left[\frac{x^3}{3} \right]_0^3$$

$$= \frac{27}{3}$$

$$= 9$$

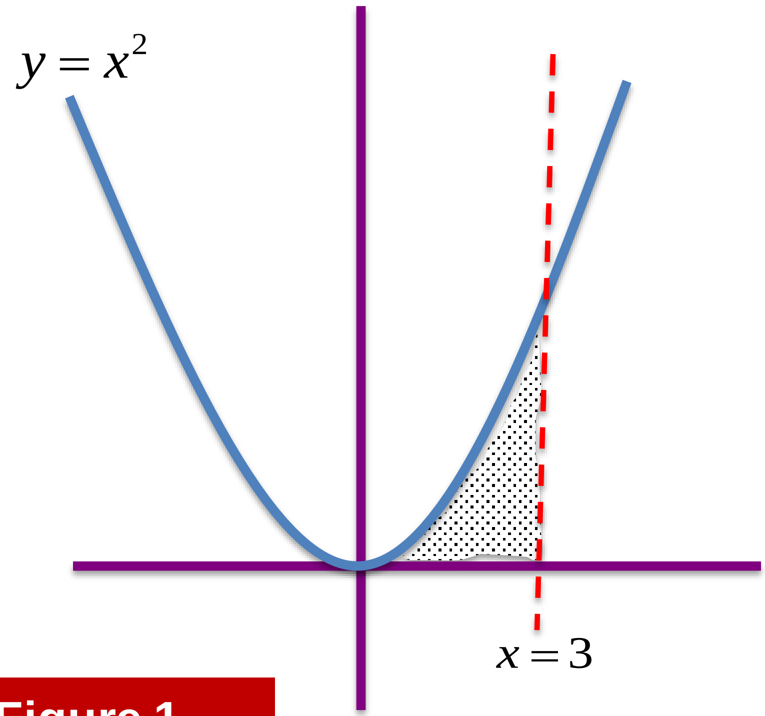


Figure 1

Example

Find the area bounded by the lines $y = 2 - x, x = 3, x = 4$ and x-axis .

The required region is the shaded region as shown in Figure 2. The shaded region is below the x -axis, therefore the area of this region is

$$\left|\int_3^4 (2-x)\,dx\right| = \left|\left[2x - \frac{x^2}{2}\right]_3^4\right|$$

$$= \left|-\frac{3}{2}\right|$$

$$= 1\frac{1}{2} \text{ unit}^2$$

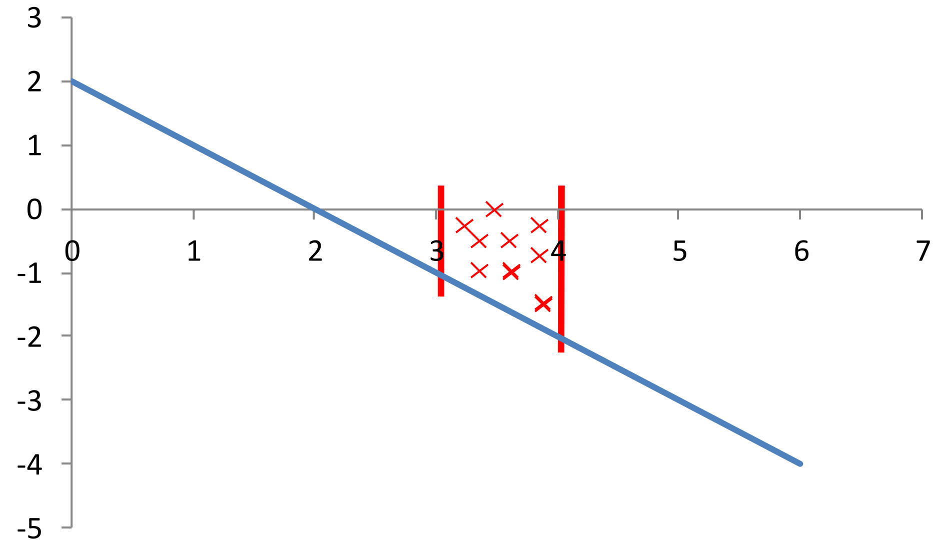


Figure 2

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Example

Find the area of the region bounded by the curve $x=\dfrac{2}{y}, y=2, y=4$ and y-axis.

The required region is the shaded region as shown in Figure 3. The area of this region is

$$\int_2^4 \frac{2}{y}\,dy = \left[2\ln y\right]_2^4$$

$$= 2\left(\ln 4 - \ln 2\right)$$

$$= 1.386 \text{ unit}^2$$

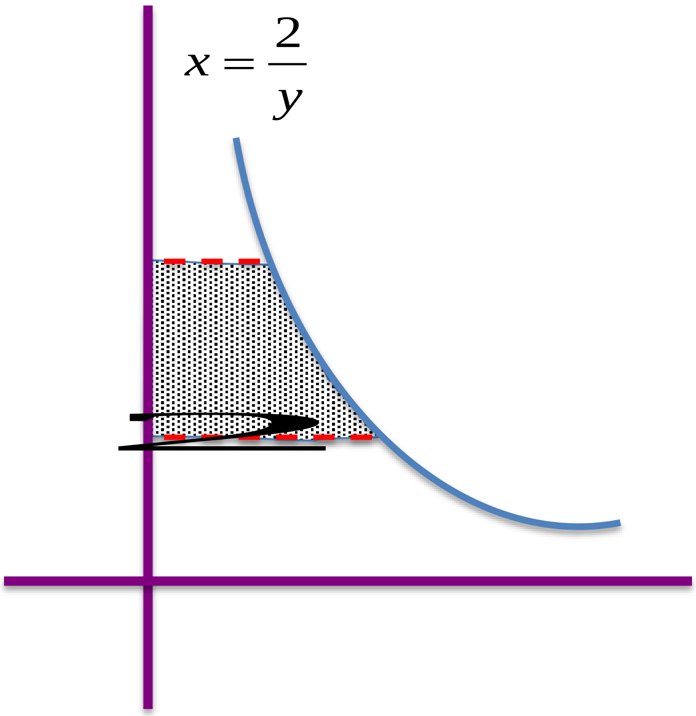


Figure 3

Example

Determine the area of the region bounded by the curve $y = x^3$ and x-axis for $-1 \le x \le 1$.

For $-1 \le x \le 0$, region is located below the x-axis whereas for $0 \le x \le 1$ the region is located above the x-axis. Area is partly above and below the x-axis

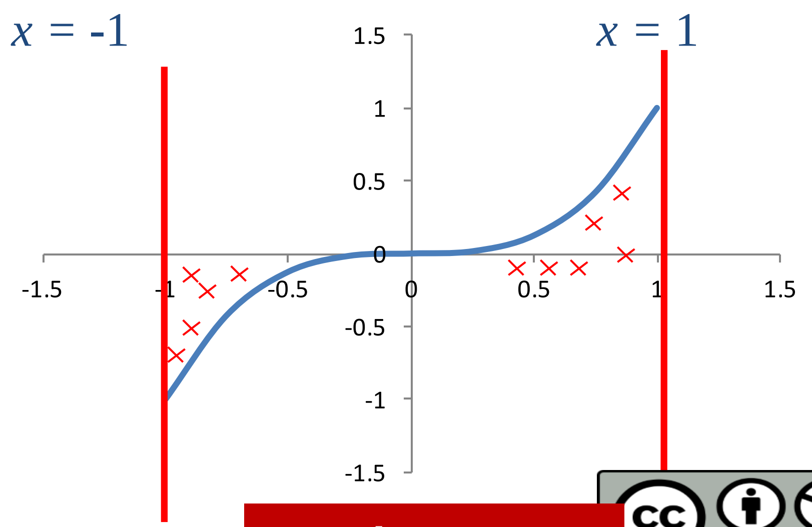


Figure 4

Area below x-axis

Area above x-axis

$$\int_{-1}^{1} x^3 dx = \left| \int_{-1}^{0} x^3 dx \right| + \int_{0}^{1} x^3 dx$$

$$= \left| \left[\frac{x^4}{4} \right]_{-1}^{0} \right| + \left[\frac{x^4}{4} \right]_{0}^{1}$$

$$= \frac{1}{2} \text{ unit}^2$$

Example

Find the total area bounded by the curve $y = x(x-1)(x-3)$ and x-axis for $0 \le x \le 3.$

For $0 \le x \le 1$, region is located above the x-axis whereas for $1 \le x \le 3$ the region is located below the x-axis.

$$\int_0^3 x(x-1)(x-3)dx$$

$$= \int_0^1 x(x-1)(x-3)dx + \left| \int_1^3 x(x-1)(x-3)dx \right|$$

$$= \int_0^1 \left(x^3 - 4x^2 + 3x\right) dx + \left| \int_1^3 \left(x^3 - 4x^2 + 3x\right) dx \right|$$

$$= \frac{37}{12} \text{ unit}^2$$

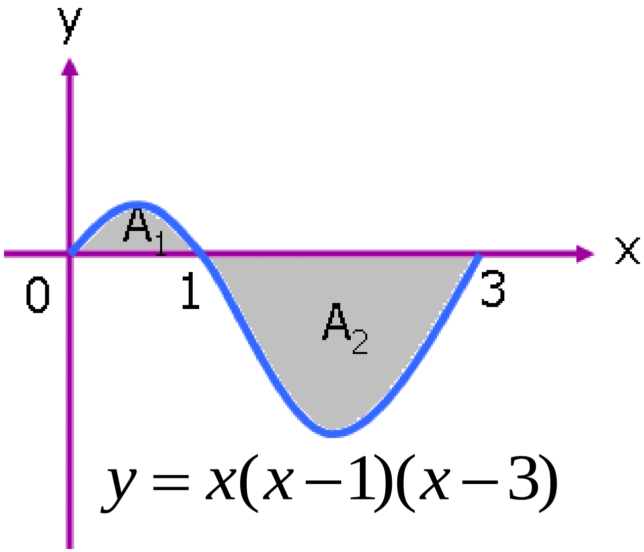


Figure 5

Example

Find the area between x-axis and the graph $f(x) = x^2 - 1$ from $x = 0$ to $x = 2$

x – intercept :

$$x^2 - 1 = 0$$
$$x = -1, 1$$

y- intercept: $y = -1$

$$A = \int_0^1 0 - (x^2 - 1)dx + \int_1^2 (x^2 - 1) - 0dx$$

$$= -\frac{x^3}{3} + x\Big|_0^1 + \frac{x^3}{3} + x\Big|_1^2$$

$$= \frac{2}{3} + \frac{4}{3}$$

$$= 2$$

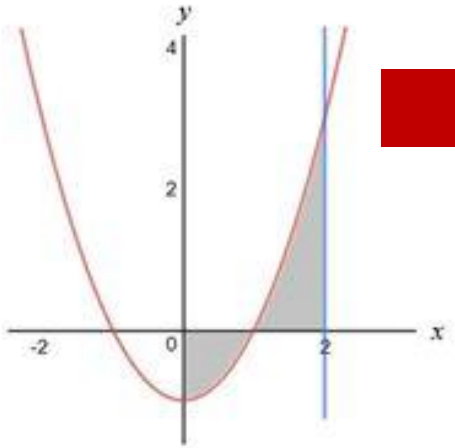


Figure 6

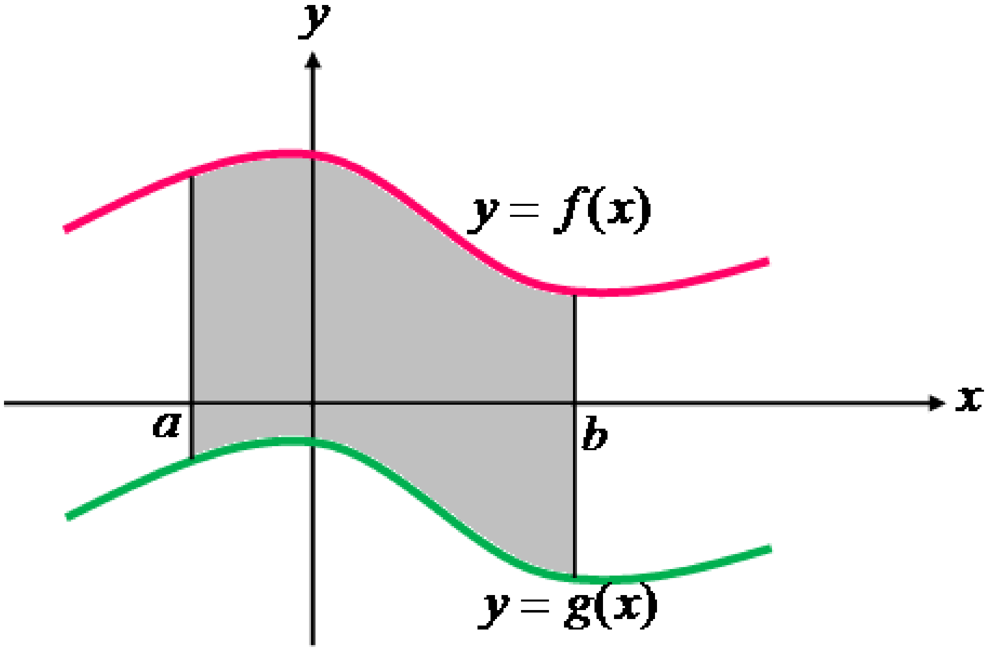
Remarks: A common mistake is to work this problem by evaluating the integral

$$A = \int_0^2 x^2 - 1dx = \frac{x^3}{3} - x\Big|_0^2 = \frac{2}{3}$$

Area Between Curves

- In some practical problems, we need to compute the area between any two curves.
- Let $f(x)$ and $g(x)$ are functions such that $f(x) \geq g(x)$ on the interval $[a, b]$.
- To find the area of the region between the two curves from $x = a$ to $x = b$, we subtract the area between the lower curve $g(x)$ and the *x*-axis from the area between the upper curve and the *x*-axis.

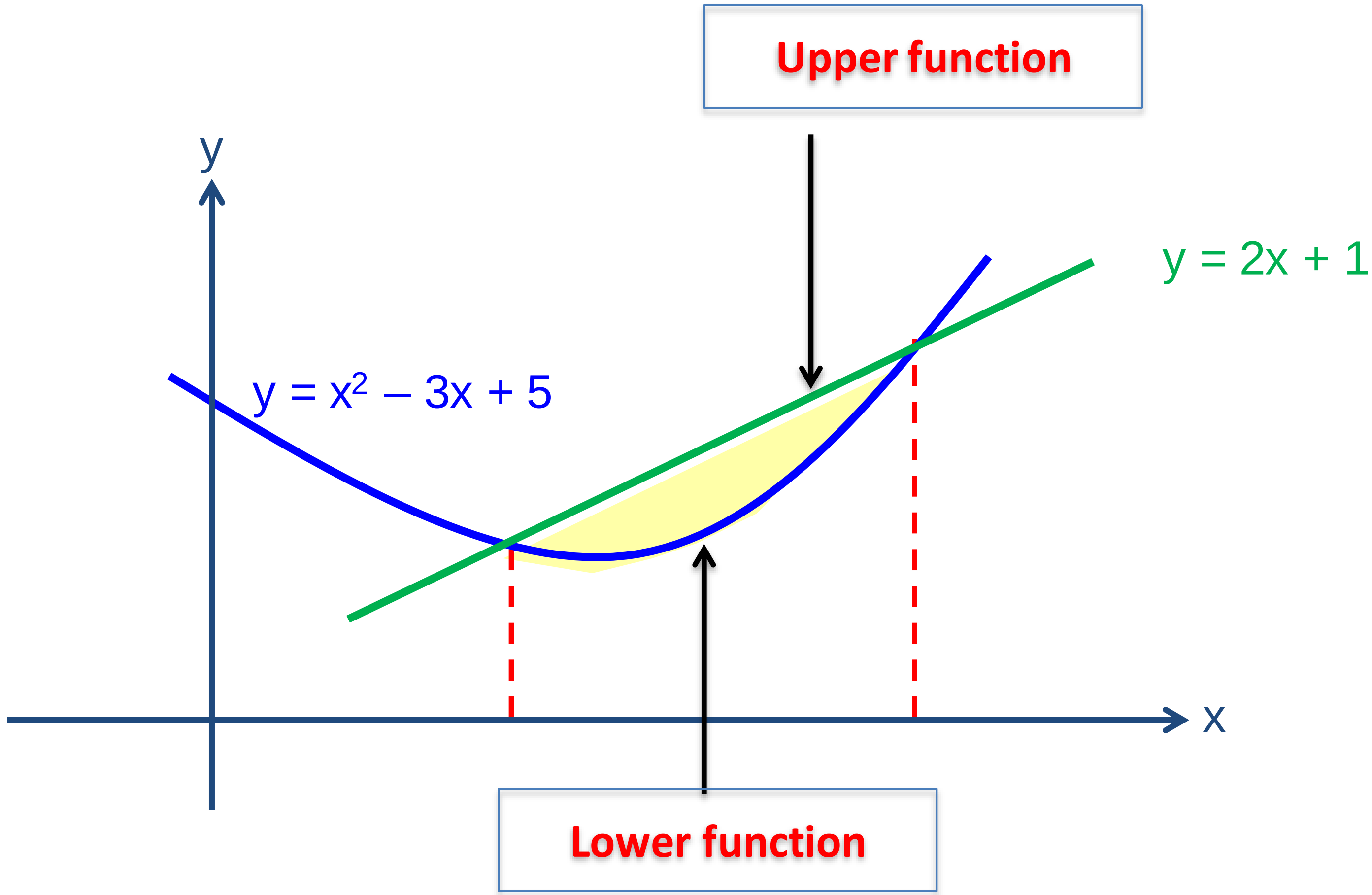
Area Between Two Curves



$$\text{Area, A} = \int_a^b \left[f(x) - g(x) \right] dx$$

Example

Find the area of the shaded region



Solution:

We need to find the upper and lower limits of the integrals by searching for the intersection points of the curve. This can be done by solving the simultaneous equations.

$$x^2 - 3x + 5 = 2x + 1$$
$$x^2 - 5x + 4 = 0$$
$$(x-4)(x-1) = 0$$

The lower limit is $a = 1$ and the upper limit is $b = 4$. Therefore, the area of the shaded region is

$$\text{Area} = \int_1^4 (2x+1) - \left(x^2 - 3x + 5\right) dx$$
$$= \left[\frac{5x^2}{2} - 4x - \frac{x^3}{3}\right]_1^4$$
$$= \frac{9}{2}$$

Example

Find the area of the shaded region by the curve $y = x^2 + 2$ and the lines $y = -x, x = 0$ and $x = 1$.

The upper and lower functions are given by $y = x^2 + 2$ and $y = -x$ respectively. The area of the shaded area is

$$\text{Area} = \int_0^1 (x^2 + 2) - (-x)\, dx$$

$$= \int_0^1 x^2 + 2 + x\, dx$$

$$= \left[\frac{x^3}{3} + 2x + \frac{x^2}{2} \right]_0^1 = \frac{17}{6}$$

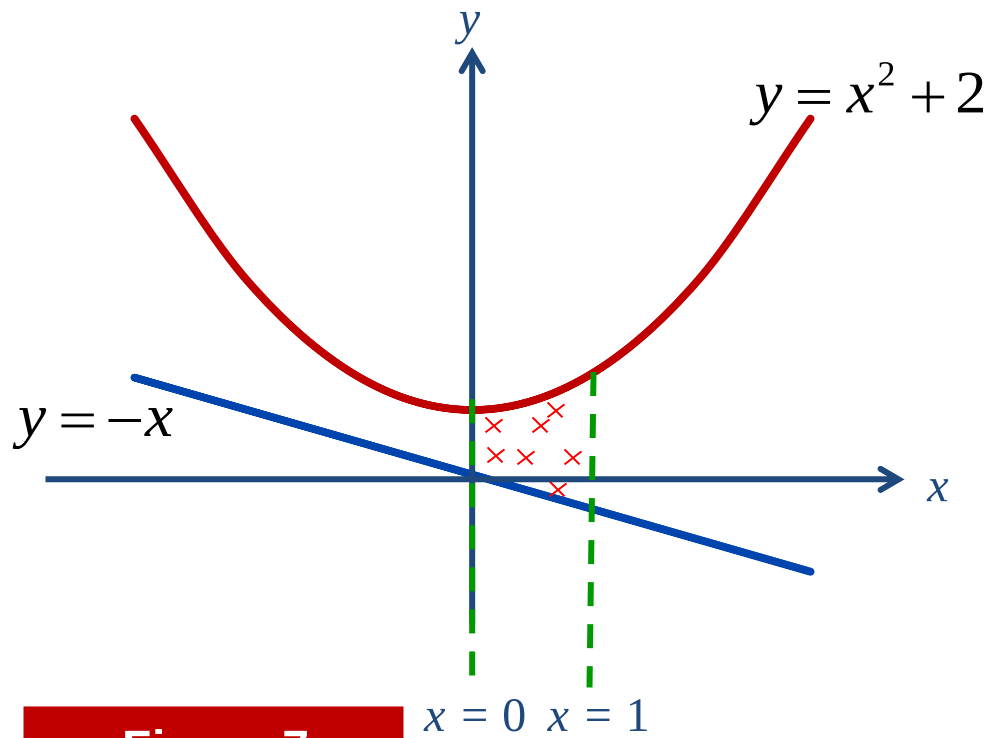


Figure 7

Example

Find the area of the region that is enclosed between the curve $y = x^2$ and the lines $y = x + 6$.

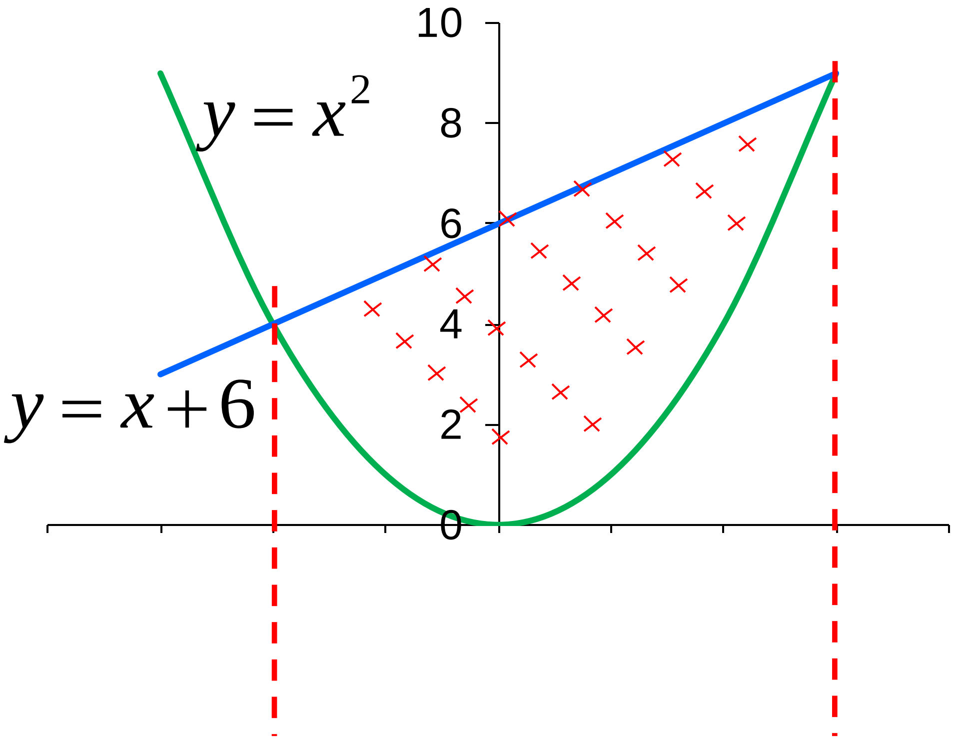


Figure 8

By solving simultaneous equations, the lower limit is $a = -2$ and the upper limit is $b = 3$. The shaded region is in interval [-2,3] and has lower function $y = x^2$ and the upper function $y = x + 6$.

$$\text{Area} = \int_{-2}^{3} (x+6) - \left(x^2\right) dx$$

$$= 20.8333 \text{ unit}^2$$

Example

Find the area of the shaded region by the curve $y = x^3$ and the lines $y = x$.

We obtain the intersection points of

$y = x^3$ and $y = x$ by solving the equation.

$$x^3 = x$$
$$x^3 - x = 0$$
$$x(x^2 - 1) = 0$$

Thus $x = 0$ and $x = \pm 1$.

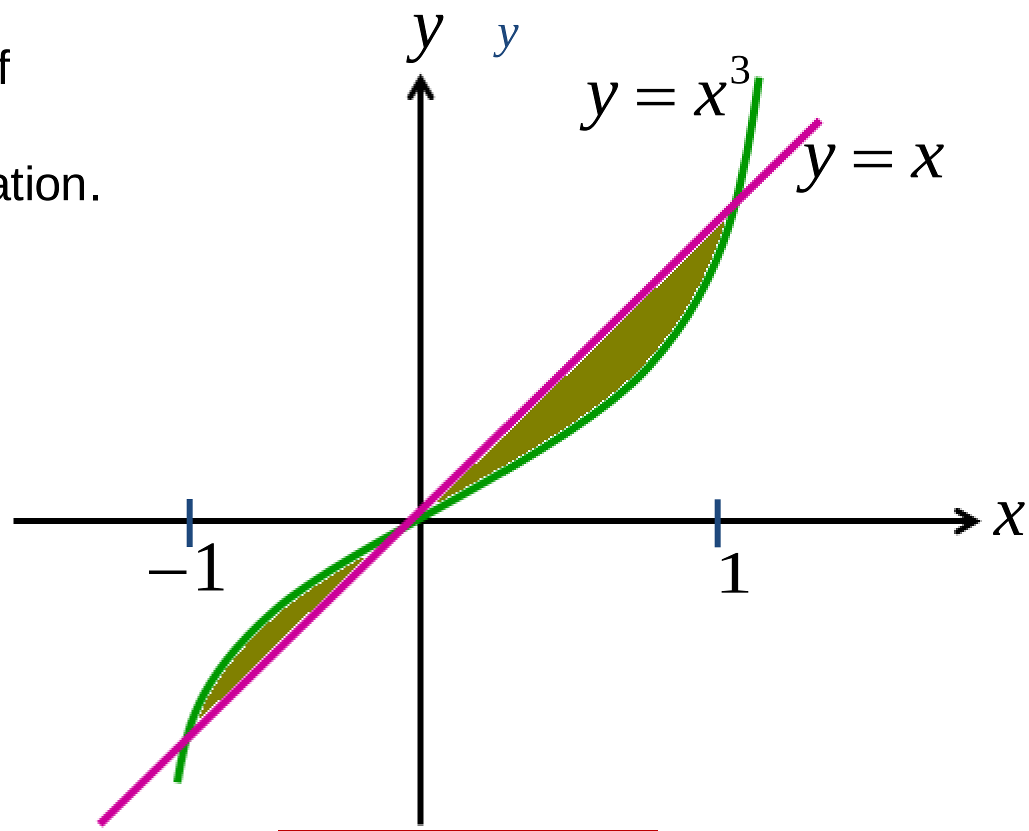


Figure 9

For $-1 \le x \le 0$, the upper function $y = x^3$ and the lower function $y = x$.

Whereas for $0 \le x \le 1$, the upper function $y = x$ and the lower function $y = x^3$.

Hence the area of the shaded region is

$$
\begin{aligned}
\text{Area} &= A_1 + A_2 \\
&= \int_{-1}^{0} \left(x^3 - x\right) dx + \int_{0}^{1} \left(x - x^3\right) dx \\
&= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} \\
&= \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2} \text{ unit}^2
\end{aligned}
$$

Example

Find the area of the region that bounded by the curve $y = 2\sqrt{x}$ and the line $y = x - 3$ in the first quadrant.

The required region is shown in shaded region in **Figure 10**. Notice that the shaded region can be obtained by subtracting the whole area with the triangle. The area is

$$\text{Area } \Delta = \int_0^9 2\sqrt{x}\, dx = \left[\frac{4}{3}x^{\frac{3}{2}}\right]_0^9 = 36$$

$$\text{Area } \Delta = \frac{1}{2}(9-3)\cdot 6 = 18$$

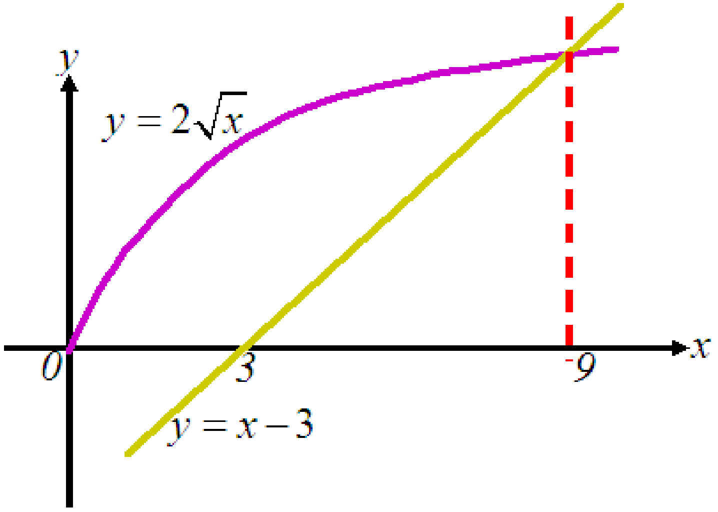


Figure 10

$$\text{Area} = 36 - 18 = 18\text{unit}^2$$

Arc Length

Definition – Arc Length: If the function $y = f(x)$ is a smooth curve on the interval $[a,b]$ then the arc length L of this curve over $[a,b]$ is defined as

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx$$

.

Example

Find the arc length (rounded to 4 decimal places) of the curve $y = x^{\frac{3}{2}}$ on the interval $[1,3]$.

Let $f(x) = x^{\frac{3}{2}}$. Therefore $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$

$$L = \int_1^3 \sqrt{1+\left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx$$

$$= \int_1^3 \sqrt{1+\frac{9}{4}x}\,dx$$

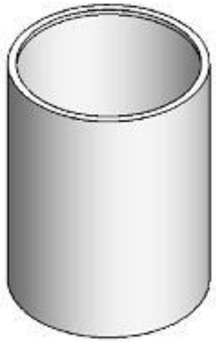
By applying substitution technique, let $u = 1+\frac{9}{4}x$. Thus $\frac{du}{dx} = \frac{9}{4}$

$$= \frac{4}{9}\int_{\frac{13}{4}}^{\frac{31}{4}} u^{\frac{1}{2}}\,du$$

$$= 4.6566$$

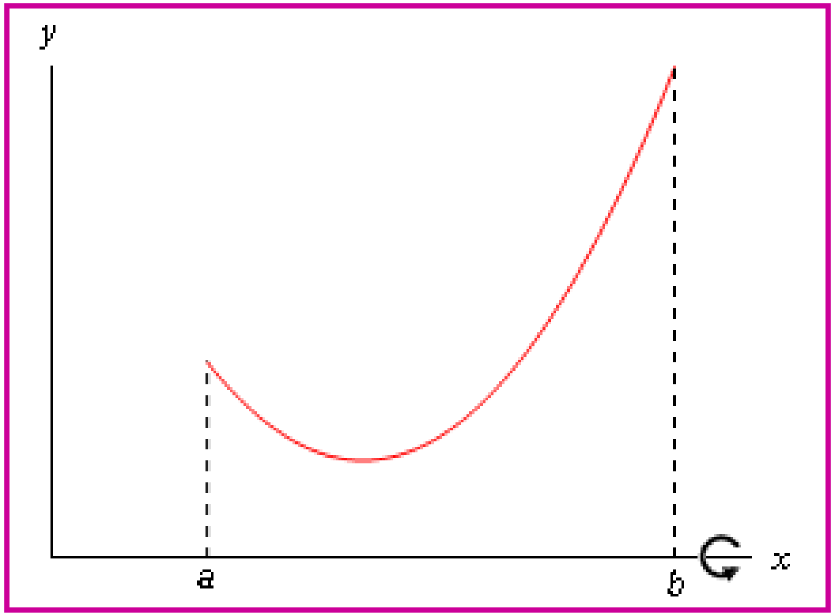
Surface Area

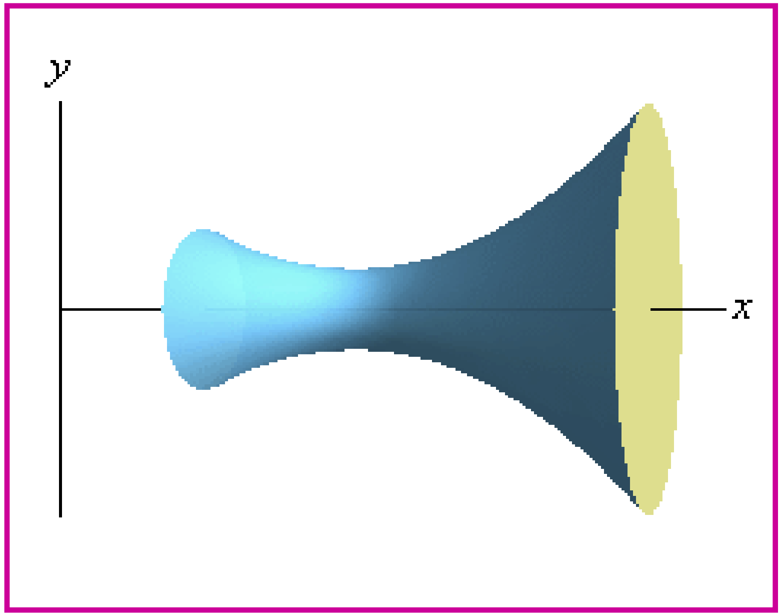
- When the arc of a curve is revolved about a line, it generates a surface.
- A surface of revolution is a surface generated by rotating a two-dimensional curve about the x-axis and y-axis. The resulting surface always has azimuthal symmetry. Examples of surface revolution include cone and cylinder.



Example

The curve $y = f(x)$ in the interval $[a,b]$ is revolved about the x-axis.





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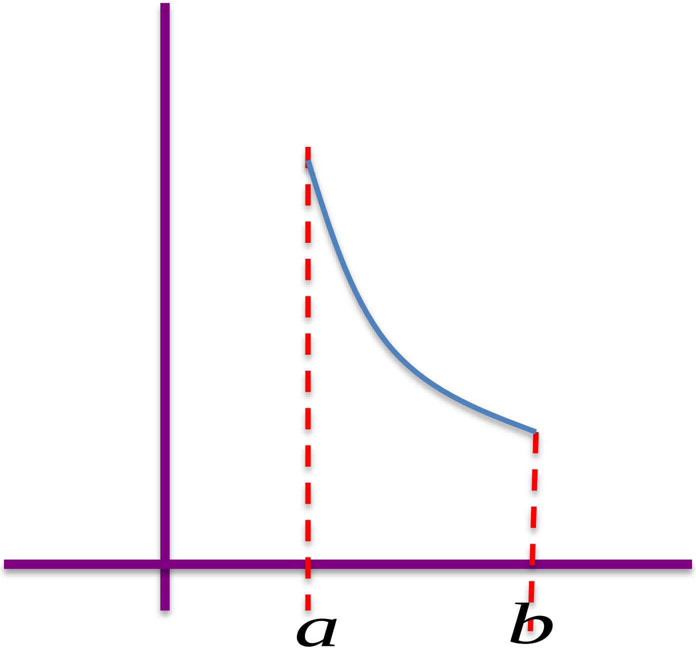
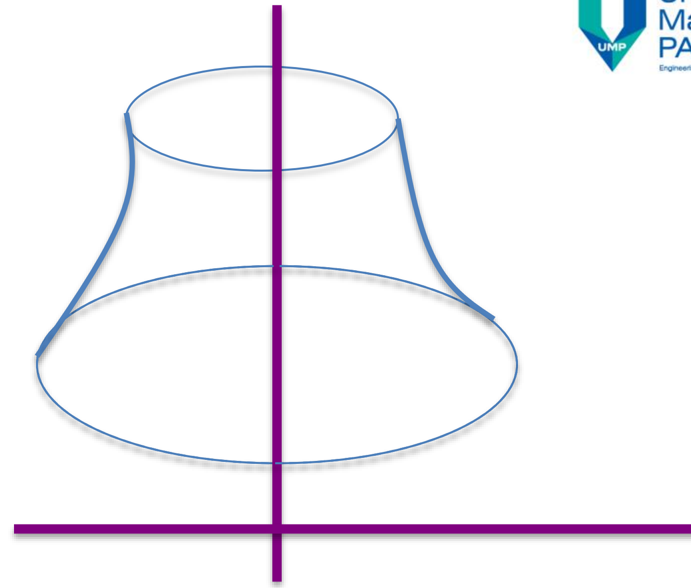
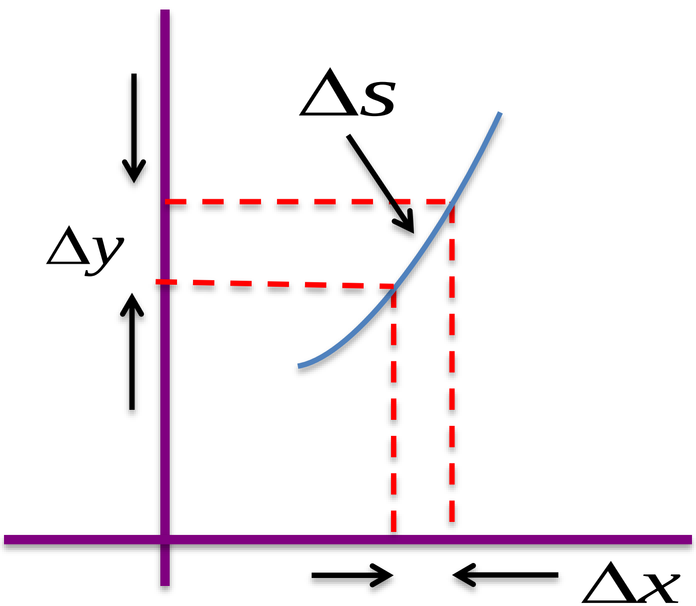
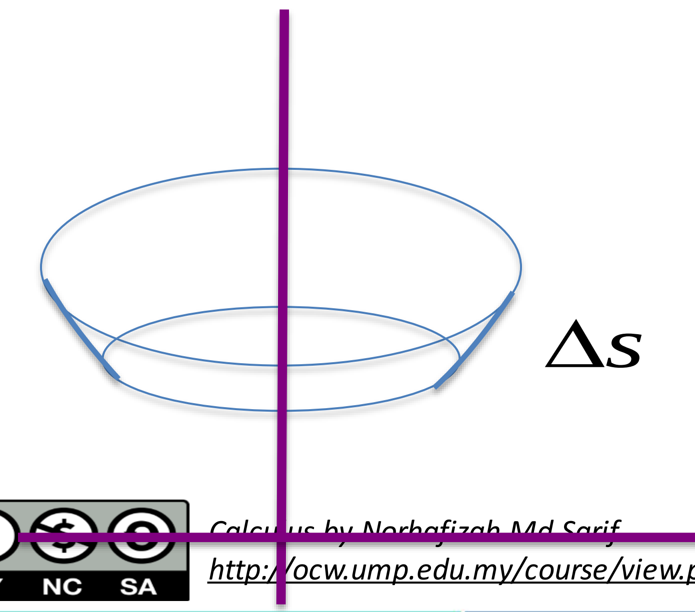
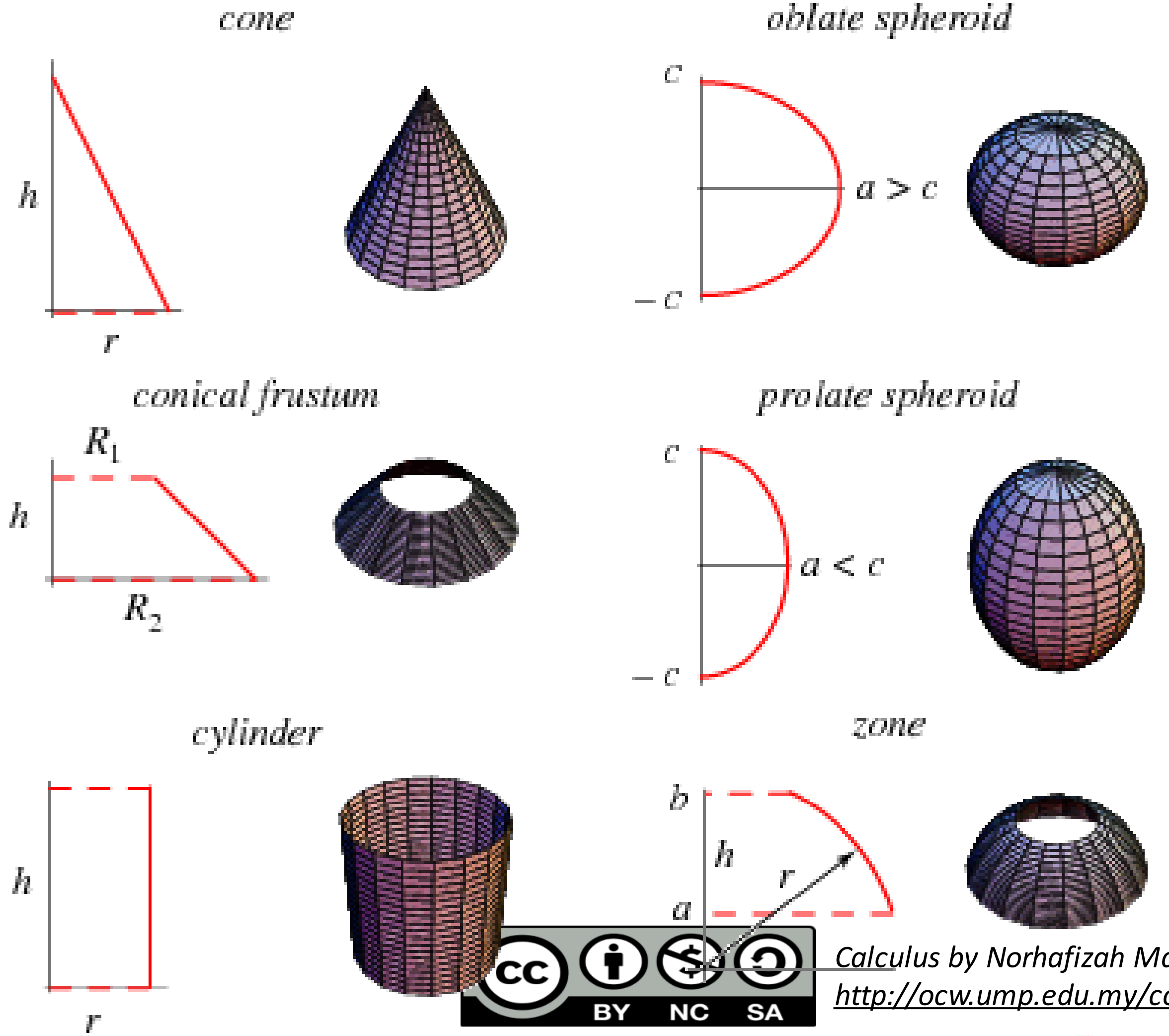








Geometry


cone
h
r
oblate spheroid
c
a > c
− c
conical frustum
R_1
h
R_2
prolate spheroid
c
a < c
− c
cylinder
h
r
zone
b
h
r
a

Surface Area

Definition – Revolving about *x-axis*: If the function $y = f(x)$ has a continuous first derivative throughout the interval $a \le x \le b$, then the area of the surface generated by revolving the curve about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

.

Surface Area

Definition – Revolving about y-*axis*: If the function $x = g(y)$ has a continuous first derivative throughout the interval $c \le y \le d$, then the area of the surface generated by revolving the curve about the y-axis is

$$S = \int_c^d 2\pi g(y)\sqrt{1+\left[g'(y)\right]^2}\,dy$$

.

Example

Find the surface area that is generated by revolving the portion of the curve $y = x^3$ between $0 \le x \le 2$, about x -axis.

By inserting the information given into the formula, we obtained

$$S = \int_0^1 2\pi \left(x^3\right)\sqrt{1+\left(3x^2\right)^2}\,dx$$
$$= \int_0^1 2\pi x^3 \sqrt{1+9x^4}\,dx$$

By applying the techniques of substitution, where

$$u = \left(1+9x^4\right),\ \frac{du}{dx} = 36x^3$$

When $x = 0,\ u = 1$ and $x = 2,\ u = 145$

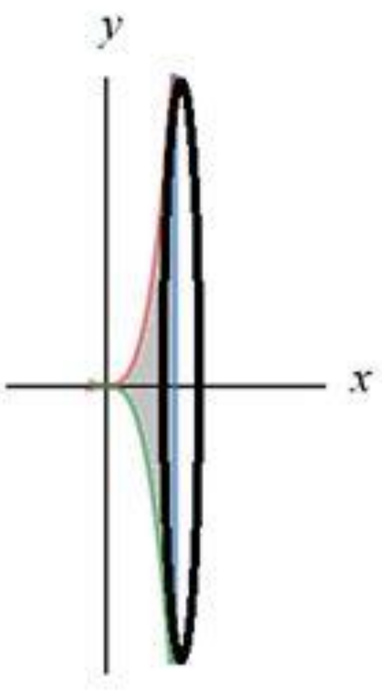


Figure 11

Simplify whenever necessary,

$$S = 2\pi \int_1^{145} x^3 \sqrt{u}\, \frac{du}{36x^3}$$

$$= \frac{\pi}{18} \int_1^{145} \sqrt{u}\, du$$

$$= \left[\frac{2\pi}{27} u^{\frac{3}{2}} \right]_1^{145}$$

$$= \frac{\pi}{27}(1746.0312 - 1)$$

$$= 64.6308\pi$$

Example

Find the surface area that is generated by revolving the portion of the curve $y=\sqrt{x}$ between $1 \le x \le 4$, about x-axis.

Surface area are given as follow

$$S=\int_1^4 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx$$

$$=\int_1^4 \pi\sqrt{4x+1}\,dx$$

By applying the techniques of substitution, where

$$u=4x+1,\quad \frac{du}{dx}=4$$

We get

$$S=\int_5^{17}\frac{1}{4}\pi\sqrt{u}\,du$$

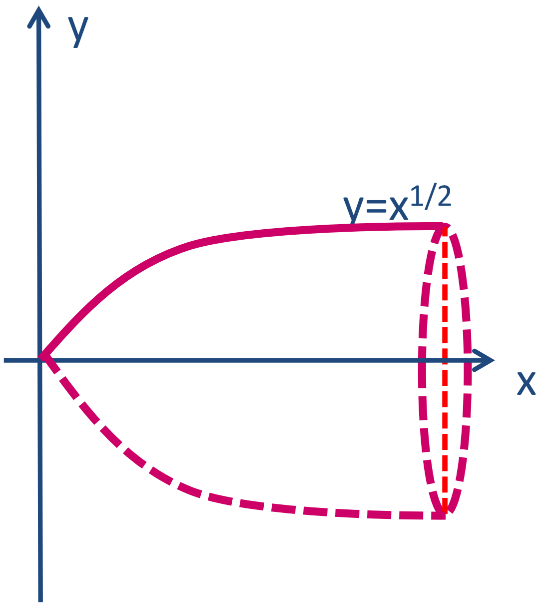


Figure 12

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