Calculus
Integration

By

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Description

Aims

This chapter is aimed to :

1. introduce the concept of integration
2. explain the basic properties of integral
3. compute the integral using different techniques of integration

Expected Outcomes

1. Students should be able to explain about indefinite integral and definite integral
2. Students should be able to know the basic properties of definite integrals
3. Student should be able to determine the appropriate techniques to solve difficult integral.

References

1. Abdul Wahid Md Raji, Hamisan Rahmat, Ismail Kamis, Mohd Nor Mohamad, Ong Chee Tiong***. The First Course of Calculus for Science & Engineering Students***, Second Edition, UTM 2016.

Content

1. Integration by Substitution
2. Integration by Parts
3. Integration using Partial Fractions

Techniques of Integration

1 Integration by Substitution

2 Integration by Parts

3 Integration by Partial Fraction

Integration by Substitution

The idea of integration by substitution is to transform a difficult integral to an simpler integral by using a substitution.

Theorem – Integration by substitution. Let f, g and u be differentiable functions of x such that

$$f(x) = g(u)\frac{du}{dx}$$

Then

$$\int f(x)dx = \int g(u)\frac{du}{dx}dx = \int g(u)du = G(u) + c$$

Integration by Substitution – Working Steps

01 Choose a new variable u. Let say $u = f(x)$

02 Differentiate u with respect to x. $\dfrac{du}{dx} = f'(x)$

03 Determine the value dx. $dx = \dfrac{du}{f'(x)}$

04

Make the substitution $u = f(x)$ and $dx = \dfrac{du}{f'(x)}$

05

Evaluate the resulting integral

06

Return to the initial variable.

Example

Evaluate $\int (5x+3)^7 \, dx$

Step 1 : Choose a substitution function u $\Rightarrow$ $u = 5x+3$

Step 2: Differentiate u with respect to x $\Rightarrow$ $\dfrac{du}{dx} = 5$

Step 3: From step 2, dx $\Rightarrow$ $dx = \dfrac{du}{5}$

Step 4: Substitute u and dx into integral $\Rightarrow$ $\int (5x+3)^7 \, dx = \int u^7 \dfrac{du}{5}$

Step 5 : Evaluate the resulting integral $\Rightarrow$ $\int \dfrac{1}{5} u^7 du = \dfrac{1}{40} u^8 + c$

Step 6 : Return to variable x $\Rightarrow$ $\dfrac{1}{40} u^8 + c = \dfrac{1}{40}(5x+3)^8 + c$

Therefore, the solution is

$$\boxed{\int (5x+3)^7 \, dx = \frac{1}{40}(5x+3)^8 + c}$$

Example

Evaluate $\int_0^{\frac{\pi}{2}} \cos x \sin^2 x dx$

Step 1 : Choose a substitution function u $\Rightarrow$ $u = \sin x$

Step 2: Differentiate u with respect to x $\Rightarrow$ $\frac{du}{dx} = \cos x$

Step 3: From step 2, dx $\Rightarrow$ $dx = \frac{du}{\cos x}$

Step 4: Substitute u and dx into integral $\Rightarrow$ $\int_0^{\frac{\pi}{2}} \cos x \sin^2 dx = \int_0^1 u^2 \cos x \frac{du}{\cos x}$

Step 5 : Evaluate the resulting integral $\Rightarrow$ $\int_0^1 u^2 du = \left.\frac{u^3}{3}\right|_0^1$

Step 6 : Compute values of u $\Rightarrow$ $\left.\frac{u^3}{3}\right|_0^1 = \frac{1}{3}$

Therefore, the solution is $\boxed{\int_0^{\pi/2} \cos x \sin^2 x dx = \frac{1}{3}}$

Example

Evaluate $\int \left(3x^2+1\right)^{30} \cdot 2x\,dx$

Let $u = 3x^2 + 1$, then $\dfrac{du}{dx} = 6x$ which implies $dx = \dfrac{du}{6x}$. The given integral can be written as

$$\int \left(3x^2+1\right)^{30} \cdot 2x\,dx = \int u^{30} \cdot 2x \frac{du}{dx}$$

$$= \frac{1}{3}\int u^{30}\,du$$

$$= \frac{1}{3}\frac{u^{31}}{31} + c$$

$$= \frac{1}{93}\left(3x^2+1\right)^{31} + c$$

Example

Evaluate $\int_{0}^{2}(5x-1)^3\,dx$

Let $u=5x-1$, then $\dfrac{du}{dx}=5$ which implies $dx=\dfrac{du}{5}$. The given integral can be written as

Method I :

When x=0, u=-1 and when x=2, u=9.

$$\int_0^2 (5x-1)^3\,dx=\int_{-1}^{9}u^3\,\frac{du}{5}$$

$$=\frac{1}{5}\frac{u^4}{4}\bigg|_{-1}^{9}$$

$$=\frac{1}{20}\left[9^4-(-1)^4\right]$$

$$=\frac{1}{20}(6561-1)=328$$

Method II :

$$\int_0^2 (5x-1)^3\,dx=\int u^3\,\frac{du}{5}$$

$$=\frac{1}{5}\frac{u^4}{4}+c$$

$$=\frac{1}{20}(5x-1)^4\bigg|_0^2$$

$$=\frac{1}{20}(6561-1)$$

$$=328$$

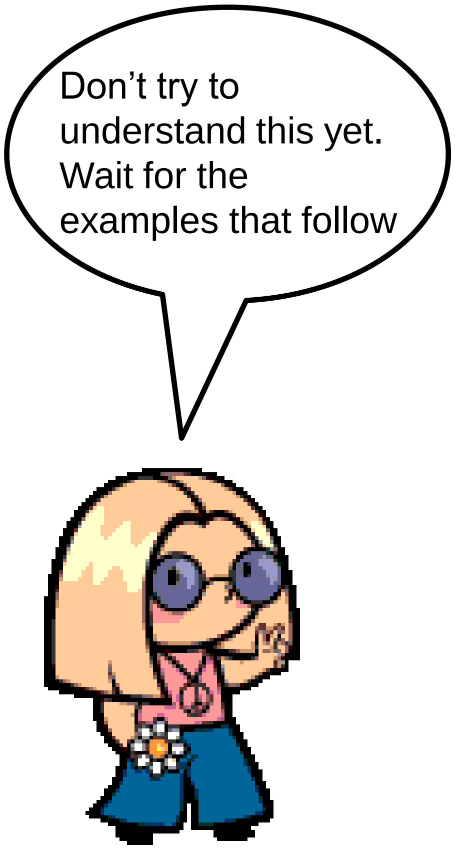
Integration by Parts

Integration by parts is a technique to solve an integration in the form of product of two functions such as:

$$\int [x^2 \sin(5x)]\,dx$$

$$f(x) = x^2 \qquad g(x) = \sin(5x)$$

The main interest in integration by parts is to transform an integral into a new integral that is easier to solve than the original.



Indefinite integrals:

$$\int u\left(\frac{dv}{dx}\right)dx = uv - \int v\left(\frac{du}{dx}\right)dx$$

Definite integrals:

$$\int_a^b u\left(\frac{dv}{dx}\right)dx = \left[uv\right]_a^b - \int_a^b v\left(\frac{du}{dx}\right)dx$$

For convenience, this can be memorized as:

$$\int u\,dv = uv - \int v\,du$$

Integration by Parts – Guideline of Selecting U

Choose u by the following sequence:

1. **L**ogarithmic (log x, lnx)
2. **A**lgebraic $(x, x^2\ x^3,\ldots)$
3. **T**rigonometry (sin x, cos x, tan x,…)
4. **E**xponential $(e^x, e^{4x},\ldots)$

and the next function automatically becomes dv.

If the new integral is more difficult than the original, change the choice of *u* and *dv*.

Example

Evaluate $\int x \cos x dx$

$$\int \underbrace{x}_{u} \underbrace{\cos x \, dx}_{dv}$$

Why choose *x* as u instead of cos x?

x is algebraic function. Meanwhile, cos x is a trigonometric function. Hence, algebraic function comes first before trigonometric functions. So x is chosen as u.

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Differentiate u

$$u = x, \qquad dv = \cos x dx$$

$$\frac{du}{dx} = 1, \qquad v = \sin x \text{ (omit c)}$$

Integrate dv

Plug everything into the formula

$$\int u dv = uv - \int v du$$
$$= x \sin x - \int \sin x dx$$

Integrating $\int \sin x dx = -\cos x + c$. Therefore,

$$\int x \cos x dx = x \sin x - \int \sin x dx$$
$$= x \sin x - (-\cos x) + c$$
$$= x \sin x + \cos x + c$$

Example

Evaluate $\int 3x^2 \ln x dx$

$$\int \underbrace{\ln x}_{u} \underbrace{3x^2 dx}_{dv}$$

Why ln x is a u?

In guideline of choosing ‘u’, we refer **LATE** in which Logarithmic (L) function comes first in the list. Hence, $\ln x$ is chosen as u

Differentiate u

$$\left[\begin{array}{ll} u = \ln x, & dv = 3x^2 dx \\ \dfrac{du}{dx} = \dfrac{1}{x}, & v = x^3 \quad \text{(omit c)} \end{array}\right]$$

Integrate dv

Plug everything into the formula

$$\int u\,dv = uv - \int v\,du$$

$$= x^3 \ln x - \int x^{\cancel{3}} \left(\frac{1}{\cancel{x}}\right) dx$$

Integrating $\int x^2 dx = \frac{1}{3}x^3 + c$, we obtained,

$$\int 3x^2 \ln x\,dx = x^3 \ln x - \int x^2 dx$$

$$= x \sin x - \frac{x^3}{3} + c$$

Example

Evaluate $\int 2xe^x \, dx$

$$u = 2x, \qquad dv = e^x dx$$

$$\frac{du}{dx} = 2, \qquad v = e^x \text{ (omit c)}$$

By using integration by parts

$$\int u dv = uv - \int v du$$

$$= 2xe^x - \int e^x (2dx)$$

$$= 2xe^x - 2e^x + c$$

- Sometimes, integration by parts must be applied several times to evaluate a given integral. See example below.

Example

Evaluate $\int e^x \cos x \, dx$

$$u = \cos x, \qquad dv = e^x dx$$

$$\frac{du}{dx} = -\sin x, \qquad v = e^x$$

By using integration by parts

$$\left.\begin{aligned}\int u\,dv &= uv - \int v\,du \\ &= e^x \cos x - \int e^x \left(-\sin x dx\right) \\ &= e^x \cos x + \int e^x \left(\sin x dx\right) + c\end{aligned}\right\} \quad 1$$

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Another integration by parts applied to the last integral, i.e.

$\int e^x \sin x\, dx$ will complete the solution. Hence, by doing by parts once again, we obtain

$$u = \sin x, \qquad dv = e^x dx$$

$$\frac{du}{dx} = \cos x, \qquad v = e^x$$

Substitute into the formula

$$\left.\begin{aligned} \int e^x \sin x dv &= uv - \int v du \\ &= e^x \sin x - \int e^x \cos x dx \end{aligned}\right\} \quad 2$$

Substitute result in (2) into Equation (1)

$$\int e^{x}\cos x dx = e^{x}\cos x + \int e^{x}\left(\sin x dx\right)$$
$$= e^{x}\cos x + \left[e^{x}\sin x - \int e^{x}\cos x dx\right]$$

Notice that the last term is similar to the original problem. Hence, by moving the last term into the left hand side equation, we get

$$\int e^{x}\cos x dx + \int e^{x}\cos x dx = e^{x}\cos x + e^{x}\sin x + c$$
$$2\int e^{x}\cos x dx = e^{x}\cos x + e^{x}\sin x + c$$

We want to find $\int e^{x}\cos x\, dx$, therefore

$$\int e^{x}\cos x dx = \frac{e^{x}\cos x + e^{x}\sin x}{2} + c$$

Integration by Partial Fractions

If the integrand is in the form of an algebraic fraction and the integral cannot be evaluated by simple methods, the fraction needs to be expressed in partial fraction.

Definition – Proper Fraction Any rational function of x,

$$\frac{P(x)}{Q(x)}$$

where the $P(x)$ is less than the degree of $Q(x)$ could be expressed as sum of relatively simpler rational functions, called partial fractions.

1. Linear Factor

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2)\ldots(a_n x + b_n)$$

Partial fraction : $\dfrac{A_1}{(a_1 x + b_1)} + \dfrac{A_2}{(a_2 x + b_2)} \cdots \dfrac{A_n}{(a_n x + b_n)}$

2. Repeated Linear Factor

$$Q(x) = (ax + b)^n$$

Partial fraction : $\dfrac{A_1}{(ax+b)} + \dfrac{A_2}{(ax+b)^2} \cdots \dfrac{A_n}{(ax+b)^n}$

3. Quadratic Factor

$$Q(x) = \left(ax^2 + bx + c\right)$$

Partial fraction : $\dfrac{Ax + B}{(ax^2 + bx + c)}$

Example

Evaluate $\int \frac{6}{(x+3)(x+1)}dx$

Step 1

Factor the denominator

$$x^2 + 4x + 3 = (x+3)(x+1)$$

Step 2

Break up the fraction into sum of "partial fractions"

$$\frac{6}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$$

Step 3

Multiply both sides of the equation by the left side denominator

$$6 = A(x+1) + B(x+3)$$

Step 4

Take the roots of the linear factors and plug them, one at a time, into *x* on the equation from step 3, and solve

If $x=-1, \quad B=3$

$x=-3, \quad A=-3$

Step 5

Split up the original integral and integrate

$$\int \frac{6}{(x+3)(x+1)}dx = \int -\frac{3}{x+3}+\frac{3}{x+1}dx$$

$$= -3\ln|x+3| + 3\ln|x+1| + c$$

Example

Evaluate $\int \frac{7x+6}{x^3-3x^2}\,dx$

The function can be written as

$$\frac{7x+6}{x^3-3x^2} = \frac{7x+6}{x^2(x-3)}$$

The denominator is a combination of linear and repeated linear case, therefore we have

$$\frac{7x+6}{x^3-3x^2} = \frac{Ax+B}{x^2} + \frac{C}{x-3}$$

We have three unknown A, B and C. Multiply both sides by the left side of denominator,

$$7x+6=(Ax+B)(x-3)+Cx^2$$
$$=Ax^2-3Ax+Bx-3B+Cx^2$$
$$=(A+C)x^2+(-3A+B)x-3B$$

Equating the coefficient of the polynomial, where

$$x^2: \quad A+C=0$$
$$x: -3A+B=7$$
$$x^0: \quad -3B=6$$

Solving the simultaneous equation, we obtain $A=-3, \quad B=-2, \quad C=3.$

Alternatively, A, B and C can be found using the another approach.

If

$$x=3, \quad 9C=27 \rightarrow C=3$$
$$x=0, \quad 6=-3B \rightarrow B=-2$$
$$x=1, \quad 13=-2(A-2)+3 \rightarrow A=-3$$

Hence,

$$\frac{7x+6}{x^3-3x^2}=\frac{-3x-2}{x^2}+\frac{3}{x-3}$$

The integration becomes

$$\int\frac{7x+6}{x^3-3x^2}dx=\int\frac{-3x-2}{x^2}+\frac{3}{x-3}\,dx$$
$$=\int\frac{-3x}{x^2}\,dx-\int\frac{2}{x^2}\,dx+\int\frac{3}{x-3}dx$$

Simplify whenever necessary and then integrate

$$\int\frac{7x+6}{x^3-3x^2}dx=\int\frac{-3}{x}\,dx-\int 2x^{-2}dx+\int\frac{3}{x-3}dx$$
$$=-3\ln|x|+\frac{2}{x}+3\ln|x-3|+c$$

Conclusion

- In integration by substitution, making appropriate choices for *u* will come with experience.
- Selecting u for by part techniques should follow the LATE guideline.
- If the power of denominator is less than the power of numerator, then the fraction is called proper fraction

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