Calculus
Integration

By

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Description

Aims

This chapter is aimed to :

1. introduce the concept of integration
2. evaluate the definite and indefinite integral
3. explain the basic properties of integral

Expected Outcomes

1. Students should be able to describe the concept of antiderivatives
2. Students should be able to explain about indefinite integral and definite integral
3. Students should be able to know the basic properties of definite integrals

References

1. Abdul Wahid Md Raji, Hamisan Rahmat, Ismail Kamis, Mohd Nor Mohamad, Ong Chee Tiong. ***The First Course of Calculus for Science & Engineering Students***, Second Edition, UTM 2016.

Content

1. Anti-derivatives
2. Indefinite Integral
3. Definite Integral

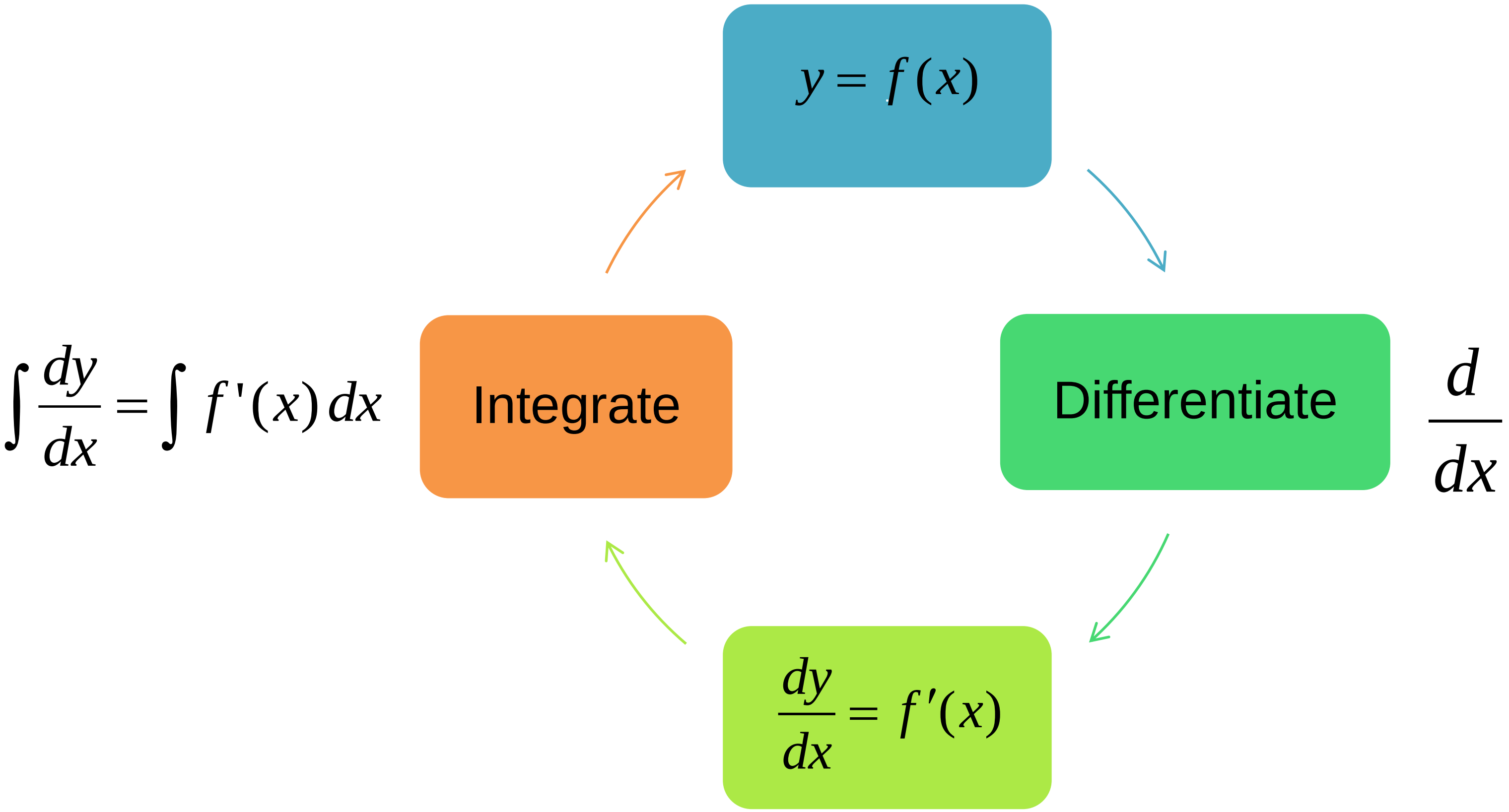
1.1 Antiderivatives

- A person who knows the acceleration of a car may want to determine its velocity or its position at a particular time.
- In this case, the derivative *f* ‘ (*x*) is given and the problem is to find the corresponding function *f*(*x*).
- The process of getting back the function *f*(*x*) from the derivative *f* ‘ (*x*) is known as **antiderivatives**.

Definition: A function F is called antiderivative of a given function f on an interval I if

$$F'(x) = f(x)$$

for all x in I.


$y = f(x)$
Differentiate
$\frac{d}{dx}$
$\frac{dy}{dx} = f'(x)$
Integrate
$\int \frac{dy}{dx} = \int f'(x)\,dx$

Example

Suppose $f(x)=2x^3$. Find a function F(x) such that $F'(x)=2x^3$

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The solutions are not unique. There are various possible solutions as given as follow

$$F(x)=\frac{1}{2}x^4$$

$$F(x)=\frac{1}{2}x^4+5$$

$$F(x)=\frac{1}{2}x^4-3$$

$$\vdots$$

Therefore, we could say that $F(x)=\frac{1}{2}x^4+c$ where c is any constant.

Theorem: If $F'(x) = f(x)$ for all x in $I,$ then every antiderivative G of f has the form

$$G(x) = F(x) + c$$

where c is a constant.

- From the theorem, in the process of finding antiderivatives, there is no unique function. Different function will be found with different constant value, c.
- The process of finding an antiderivative is known as **integration**.

$$\int f(x)dx = F(x) + c$$

$$\int 3x^2 dx = x^3 + c$$

Function

Constant of integration

Integral symbol : Derived from the stretched capital letter S for Summation

Integrate with respect to x. x is an independent variable

Example

Find antiderivatives for the given functions

$$\text{a)}\quad f(x) = x^6, \quad \text{b)}\quad s(x) = \cos x, \quad \text{c)}\quad y(x) = 7$$

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a) $\dfrac{x^7}{7} + c,$

b) $\sin x + c,$

c) $7x + c$

4.2 Indefinite Integral

Definition – Indefinite Integral: The function

$$\int f(x)dx = F(x) + c$$

where c is an arbitrary constant, means that F is an anti derivative of f. It is called the indefinite integral of f and satisfied the condition that $F'(x) = f(x)$ for all x, in the domain of f.

Table of Integration

Basic integral formula for some general functions

$\int kx\,dx = kx + C$	$\int \cos x\,dx = \sin x + C$
$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$	$\int \sec^2 x\,dx = \tan x + C$
$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int e^x\,dx = e^x + C$	$\int \sec x \tan x\,dx = \sec x + C$
$\int \sin x\,dx = -\cos x + C$	$\int \csc x \cot x\,dx = -\csc x + C$

Example

Evaluate the following indefinite integral

a) $\int -2\, dx$

b) $\int x^5\, dx$

c) $\int \frac{1}{x^3}\, dx$

d) $\int 3\sqrt{x}\, dx$

e) $\int 8e^x\, dx$

f) $\int \frac{7}{x}\, dx$

Solution:

a) $-2x + c$

b) $\frac{x^6}{6} + c$

c) $\frac{1}{2x^2} + c$

d) $2x^{\frac{3}{2}} + c$

e) $8e^x + c$

f) $7\ln x + c$

Basic Properties of Integral

Constant Multiple Rule

$$\int k\ f(x)\,dx = k\int f(x)\,dx$$

Sum & Difference Rule

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Linearity Rule

$$\int [a\ f(x) \pm b\,g(x)]\,dx = a\int f(x)\,dx \pm b\int g(x)\,dx$$

Example

Evaluate the following definite integral

a) $\int 4x^3\ dx$

b) $\int \left(1 + x^3 - \frac{1}{x^2}\right) x^2\ dx$

c) $\int -7\cos x + \frac{3}{2} x\, dx$

d) $\int -4\sqrt{x} + \tan x + 5e^x\ dx$

e) $\int 2x^5 + \sec x \tan x\, dx$

f) $\int \frac{-2x^6 + 7x^2 - 4}{x^3} dx$

g) $\int \frac{\sin x + \cos x}{2}\ dx$

h) $\int \left(x + \frac{4}{x} \right)^2 dx$

i) $\int 3 + 4e^x + \frac{2}{5x}\ dx$

j) $\int 5\sin x + \frac{1}{e^{2x}} dx$

Answers to selected questions:

a) $\int 4x^3\,dx = 4\frac{x^4}{4}+c = x^4+c$ ← Apply $\int x^n\,dx = \frac{x^{n+1}}{n+1}+C,\ n\neq -1$

b) $\int\left(1+x^3-\frac{1}{x^2}\right)x^2\,dx = \int\left(x^2+x^5-1\right)dx = \frac{x^3}{3}+\frac{x^6}{6}-x+c$

Expand first and then integrate. Apply sum and difference rule

d) $\int -4\sqrt{x}+\tan x+5e^x\,dx = -4\frac{x^{\frac{3}{2}}}{3/2}+\sec^2 x+5e^x+c = -\frac{8}{3}x^{\frac{3}{2}}+\sec^2 x+5e^x+c$

f) $\int\frac{-2x^6+7x^2-4}{x^3}\,dx = \int -2x^3+\frac{7}{x}-4x^{-3} = -\frac{x^4}{2}+7\ln|x|+\frac{2}{x^2}+c$ ← Divide each term by x^3. Integrate each term individually.

h) $\int\left(x+\frac{4}{x}\right)^2 dx = \int x^2+8+\frac{16}{x^2}\,dx = \frac{x^3}{3}+8x-\frac{16}{x}+c$

Bring x^2 up, it then becomes x^{-2}. Integrate by using power rule

$$\int [f(x)\cdot g(x)]\,dx \neq \int f(x)\,dx \cdot \int g(x)\,dx$$

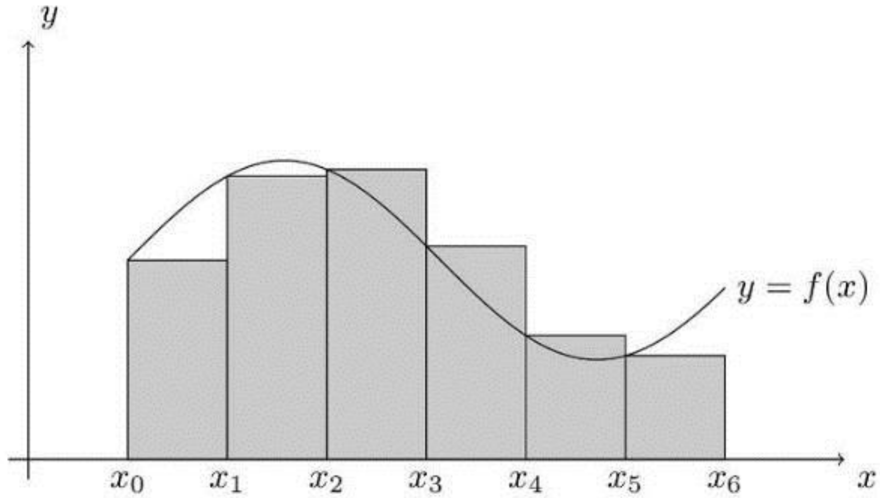
$$\int [f(x)\cdot g(x)]\,dx \neq f(x)\int g(x)\,dx$$

$$\int \frac{f(x)}{g(x)}\,dx \neq \frac{\int f(x)\,dx}{\int g(x)\,dx}$$

When dealing with the integration of product or quotient rule, we need to use some integration techniques to evaluate this kind of integral. Direct integration of each functions are strictly wrong!

4.3 Definite Integral

- The concept of definite integral links with the concept of area.
- The area can be first approximated by sums and then obtained exactly by taking a limit involving the approximating sums.
- The special kind of limit of a sum that appears in this context is called the definite integral.



Definition – Definite Integral: If f is continous on the closed interval [a,b], we say f is integrable on [a,b] if

$$\lim_{\|P\|\to 0}\sum_{k=1}^{n} f(x_k)\Delta x_k$$

exist. This limit is called the **definite integral** of f from a to b. The definite integral is denoted by

Upper limit

Lower limit

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Properties of Definite Integrals

Let k be a constant. If $f(x)$ and $g(x)$ are continuous functions on the interval $[a,b]$, then

1. $$\int_a^a f(x)\,dx = 0$$

2. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

3. $$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

4. $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a \le c \le b$$

5. $$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Example

Evaluate the following definite integral

a) $\int_{4}^{4} 2\sin 7x dx$　　b) $\int_{1}^{0} 1 - x^2 dx$

c) $\int_{0}^{2} 5 - 3\left(x^2 + 1\right) dx$　　d) $\int_{-\pi/2}^{\pi/2} 1 + \sin x dx$

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a) $\int_{4}^{4} 2\sin(7x) dx = 0$

b) $\int_{1}^{0} 1 - x^2 dx = -\int_{0}^{1} 1 - x^2 dx = -\left(x - \frac{x^3}{3}\right)\Bigg|_{0}^{1} = -\left(\frac{2}{3} - 0\right) = -\frac{2}{3}$

c) $\int_{0}^{2} 5 - 3(x^2 + 1) dx = -4$

d) $\int_{-\pi/2}^{\pi/2} \left(1 + \sin x\right) dx = \pi$

Example

Evaluate the following definite integral

a) $\int_0^2 \left(\frac{x^3}{4} + x \right) dx$

b) $\int_{-1}^3 \left[e^{-x} + 1 - \frac{2}{7x} \right] dx$

c) $\int_1^{1.5} \tan(x)\, dx$

d) $\int_0^{\pi} 8\cos x\, dx$

e) $\int_{1.2}^2 \left(\sin x - \sqrt{x^3} \right) dx$

Answers:

a) $\int_0^2 \left(\frac{x^3}{4}+x\right)dx = \frac{x^4}{16}+\frac{x^2}{2}\bigg|_0^2 = \left(\frac{3}{2}\right)-0=\frac{3}{2}$

b) $\int_{-1}^{3}\left[e^x+1-\frac{2}{7x}\right]dx = e^x + x - \frac{2}{7}\ln|x|\bigg|_{-1}^{3} = \left(e^3+3-\frac{2}{7}\ln|3|\right)-\left(e^{-1}+(-1)-\frac{2}{7}\ln|1|\right)$

$= 23.4038$

c) $\int_1^{1.5} \tan x dx = -\ln|\cos x|\bigg|_1^{1.5} = \ln\left|\frac{\cos 1}{\cos 1.5}\right| = 2.0332$

d) $\int_0^{\pi} 8\cos x dx = 8\sin x\big|_0^{\pi} = 0$

e) $\int_{1.2}^{2}\left(\sin x-\sqrt{x^3}\right) = \cos x - \frac{2}{5}x^{5/2}\bigg|_{1.2}^{2} = -2.6789+0.2686-2.4102$

Example

If $\int_{-3}^{4} f(x)dx = 11$ and $\int_{-3}^{9} f(x)dx = -5$, what is $\int_{4}^{9} f(x)dx$?

$$\int_{-3}^{9} f(x)dx = \int_{-3}^{4} f(x)dx + \int_{4}^{9} f(x)dx$$

$$-5 = 11 + \int_{4}^{9} f(x)dx$$

$$\int_{4}^{9} f(x)dx = -5 - 11$$

$$= -16$$

Example

Evaluate $\int_{-1}^{3} |x|\, dx$

$|x|$ is a piecewise function

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$$

Therefore,

$$\begin{aligned} \int_{-1}^{3} |x|\, dx &= \int_{-1}^{0} |x|\, dx + \int_{0}^{3} |x|\, dx \\ &= \int_{-1}^{0} -x dx + \int_{0}^{3} x dx \\ &= -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{3} \\ &= \frac{1}{2} + \frac{9}{2} \\ &= 5 \end{aligned}$$

Conclusion

- Integration is anti-derivative
- Any indefinite integral will have +c at the end of the solution.
- Product or quotient function cannot be integrate directly. Appropriate techniques should be used to solve this type of integral

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