Exercise 5: Applications of Differentiation

Topic 5.1 : Curve Slope at a Point

1. Find the gradient of each of the following curves at the given points

   a) $y = 2x^2 - 4x - 8; \quad x = 3$

   b) $4y = 7x^3; \quad x = 2$

   c) $y = 2x^3 + 3x^2 - 14x + 4; \quad x = 1$

   d) $y = (1 + 2x)^2; \quad x = 9$

   e) $y = \dfrac{4}{x}; \quad x = 3$

   f) $y = (-x - 2)(4 + 5x^2); \quad x = 2$

   g) $y = \dfrac{3x + 1}{-x + 4}; \quad x = 1$

   $\left[\text{a) } 8, \text{ b) } 21, \text{ c) } -2, \text{ d) } 76, \text{ e) } -\dfrac{4}{9}, \text{ f) } -104, \text{ g) } \dfrac{13}{9}\right]$

2. Find the gradient of the curve $y = 5 + 2x - x^2$ at the point where the curve meets the $y - axis$

   $[2]$

3. The gradient of the curve $y = ax^2 + b$ at the point $(2,3)$ is 8. Find the value of a and b.

   $[a = 2, \quad b = -5]$

4. Find an equation of the tangent line to the graph $f(x) = \dfrac{1}{x}$ at the point where $x = 2$

   $\left[y = -\dfrac{1}{4}x + 1\right]$

5. Find the slope of the curve $y^2 = x^3$ at the point $P(t^2, t^3)$

   $\left[\dfrac{3}{2}t\right]$

6. A curve is given by the parametric equations

$$x = t + \frac{2}{t}, \quad y = t - \frac{1}{t} \qquad t \neq 0$$

Find the gradient of the curve when $t = 4$

$$\left[\frac{17}{14}\right]$$

7. Find the slope of the curve $x^2y + xy^2 = 14$ at the point (1,3)

$$\left[-\frac{15}{7}\right]$$

8. A curve has a parametric equations

$$x = t^2 - 1, \quad y = t(t^2 + 1)$$

Find $\frac{dy}{dx}$ in terms of t

$$\left[\frac{3t^2 + 1}{2t}\right]$$

9. Find the value of $\frac{dy}{dx}$ at the point $(\frac{2}{3}r, \frac{2}{3}r)$ on the curve $x^3 + y^3 = 3xry$

$$[-1]$$

10. Find an expression in terms of x, y and q for the gradient at any point on the curve $x^3 + y^3 = 3qy^2$ with q as a constant.

$$\left[-\frac{3x^2}{3y^2 - 6qy}\right]$$

11. The parametric equations of a curve is given by

$$x = t^2 + 1, \quad y = t^3$$

Show that the point (5,-8) is a point on the curve and find the equation of tangent to the curve at that point.

$$[3x + y = 7]$$

Topic 5.2 : Rate of Change/Related Rates/Motion

12. The radius of a circle is increasing at the rate of $\frac{1}{2}\text{cms}^{-1}$. Find the rate of change of the area when the radius is 4cm.

$$\left[\frac{dA}{dt} = 4\pi\text{cm}^2\text{s}^{-1}\right]$$

13. The side of a cube is increasing at the rate of 6ms^{-1}. Find the rate of change of the volume when the length of a side is 9m.

$$\left[\frac{dV}{dt} = 1458\text{m}^2\text{s}^{-1}\right]$$

14. Oil from leaking oil tanker radiates outward in the form of circular film on the surface water. If the radius of the circle increases at the rate of 3m/min, how fast is the area of the circle increasing when the radius is 200m. *area of circle $= \pi r^2$

$$\left[\frac{dA}{dt} = 1200\pi m^2 / \min\right]$$

15. A cylindrical container has one end opened and the other end closed. It has a circular base of radius r cm. Given that the total surface area of the container is $300\pi\text{cm}^2$.
Show that the volume of the container is $V = 150\pi r - \frac{\pi r^3}{2}$

16. The volume of a spherical balloon is increasing at a constant rate of $0.05\text{m}^3/\text{s}$. Find the rate of increase of its radius when the volume of the balloon is 0.008m^3. Given, volume of sphere $= \frac{4}{3}\pi r^3$

$$\left[\frac{dr}{dt} = 0.26 cms^{-1}\right]$$

17. Water is poured into an inverted right circular cone, with radius at the top 5m and depth 10m at a rate of $8\text{ m}^3\text{s}^{-1}$. Find the rate at which depth of the water is increasing when the depth is 4m.

$$\left[\frac{dh}{dt} = \frac{2}{\pi}\text{ms}^{-1}\right]$$

18. The area of a square is increasing at the rate of $8\text{ cm}^2\text{s}^{-1}$. Find the rate of increase of the length of a side when the area is 64cm^2.

$$\left[\frac{1}{2}\text{cms}^{-1}\right]$$

19. A spherical balloon is being inflated and, at the instant when its radius is 3m, its surface area is increasing at a rate of $2\text{m}^2\text{s}^{-1}$. Find the rate of increase at the same instant of a) the radius b) the volume

$$\left[\text{a) } \frac{1}{12\pi}\text{cms}^{-1}, \text{ b) } 3\text{m}^3\text{s}^{-1}\right]$$

20. A conical cup is 4cm across and 6cm deep. Water leaks out of the bottom at the rate of $2\text{cm}^3\text{s}^{-1}$. How fast is the water level dropping when the height of the water is 3cm?

$$\left[\frac{2}{\pi}\text{cms}^{-1}\right]$$

21. Air is escaping from a spherical balloon at the rate of 2cm^3 per minute. How fast is the surface area shrinking when radius is 1 cm?

$$\left[\frac{1}{3}\text{cm}^2/\text{min}\right]$$

22. An airplane is flying towards a radar station at a constant height of 6km above the ground. If the distance, s between the airplane and the radar station is decreasing at a rate 400km per hour when $s = 10\text{km}$, what is the horizontal speed of the plane?

$$[500 \text{ km per hour}]$$

23. The radius of a right circular cylinder is increasing at the rate of 4cm/s but its total surface area remains constant at $600\pi^2$. At what rate is the height changing when the radius is 10cm?

Ans: $\left[16\text{cms}^{-1}\right]$

24. Assume that sand allowed to pour onto a level surface will form a pile in the shape of a cone, with height equal to diameter of the base. If sand is poured at 2 cubic meters per second, how fast is the height of the pile increasing when the base is 8 meters in diameters?

$$\left[\frac{1}{8\pi}\text{ms}^{-1}\right]$$

25. A stone is thrown vertically upwards from a place 10meters from the ground with an initial velocity of 23ms^{-1}. After t seconds, the height of the stone from the ground is given by

$$x = -5t^2 + 23t + 10$$

(i) Find the time when the stine reaches the highest position from the ground
(ii) Calculate the maximum height
(iii) Find the time when the stone touches the ground again

$$[\text{(i) } 2.3\text{s}, \text{ (ii) } 36.45\text{m}, \text{ (iii) } t = 5\text{s}]$$

26. A particle is moving along a straight line so that its position from the fixed point A at any time t seconds is given by

$x = t^2 - 5t + 6$

Find,

(i) its initial position
(ii) its initial velocity
(iii) the first time the particle passes A and find the velocity at that time
(iv) the second time the particle passes A again and determine the velocity at that time
(v) the time and the position of the particle when the velocity is zero.

[(i) 6m, (ii) -5ms^{-1}, (iii) 2s, -1ms^{-1}, (iv) 3s, 1ms^{-1}, (v) 2.5s, -0.25m]

27. After t seconds the position of a particle which is moving along a straight line is
$x = 2t^3 - 9t^2 + 12t + 6$

(i) When is the acceleration zero? Determine the velocity at that time?
(ii) When is the velocity zero? Determine the accelaration at that time?

[(i) 1.5s, -1.5ms^{-1}, (ii) 1s or 2s, $\pm$6ms^{-2}]

28. Ladder 9 meters long leaning against a wall. The bottom of a ladder is pulled along the ground away from the wall at a constant rate of 1 meter per second. How fast will the top of the ladder be falling when the ladder is 3 meters away from the wall.?

[-0.35ms^{-1}]

29. The area A cm^2 of the image of a missile on the screen is given by A=12/r^2, where r is the distance from the missile to radar. The missile is approaching the radar at a rate of 0.5 kms^{-1}. Find the rate of change of the area of image when the missile is 10km away. If the rate of change is 0.096 cm^{2}s^{-1}, how far is the missile from the radar?

[0.012cm^{2}s^{-1}, 5km]