Calculus
Applications of Differentiation

By

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Description

Aims

This chapter is aimed to :

1. introduce the concept of integration
2. evaluate the definite and indefinite integral
3. explain the basic properties of integral
4. compute the integral using different techniques of integration

Expected Outcomes

1. Students should be able to describe the concept of antiderivatives
2. Students should be able to explain about indefinite integral and definite integral
3. Students should be able to know the basic properties of definite integrals
4. Student should be able to determine the appropriate techniques to solve difficult integral.

References

1. Abdul Wahid Md Raji, Hamisan Rahmat, Ismail Kamis, Mohd Nor Mohamad, Ong Chee Tiong***. The First Course of Calculus for Science & Engineering Students***, Second Edition, UTM 2016.

Content

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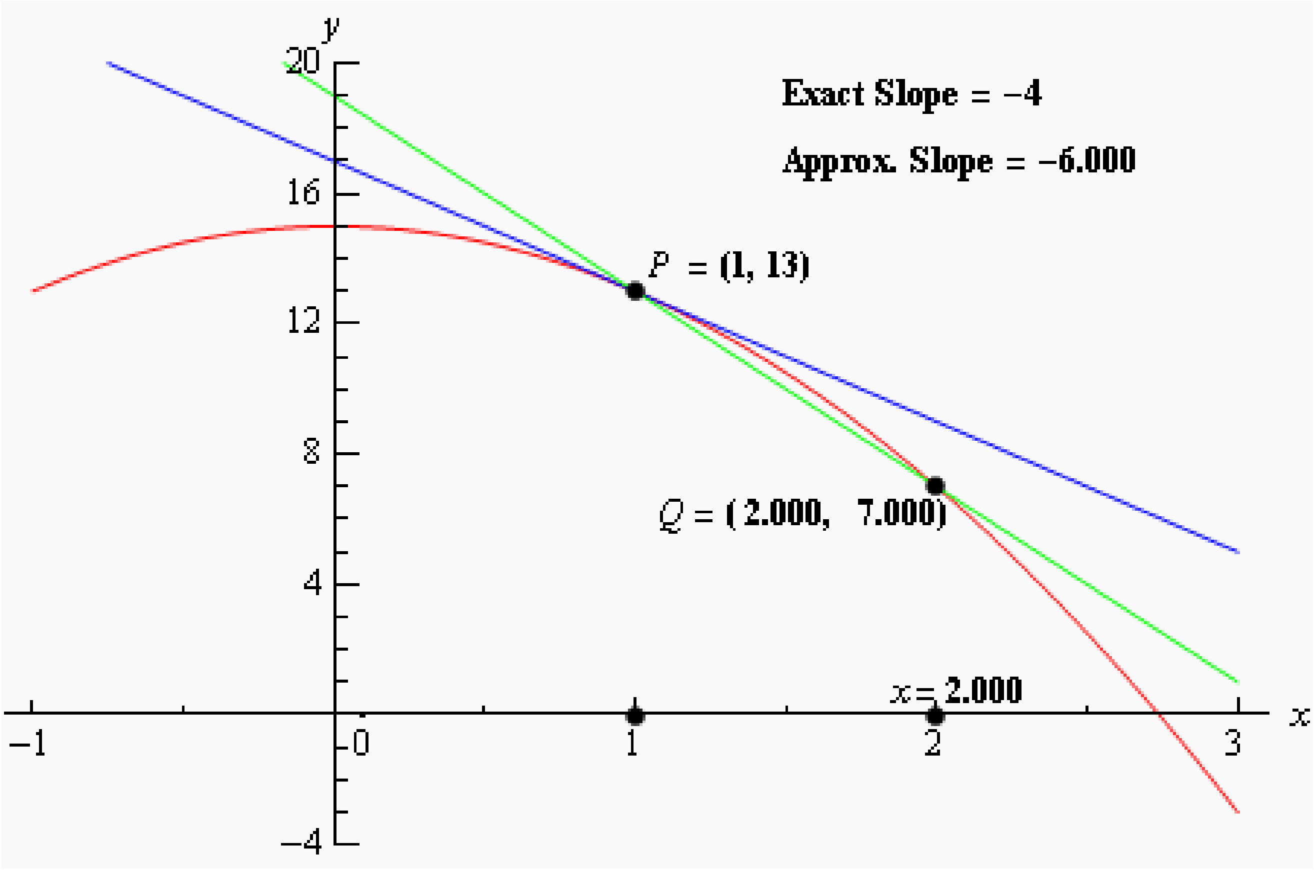
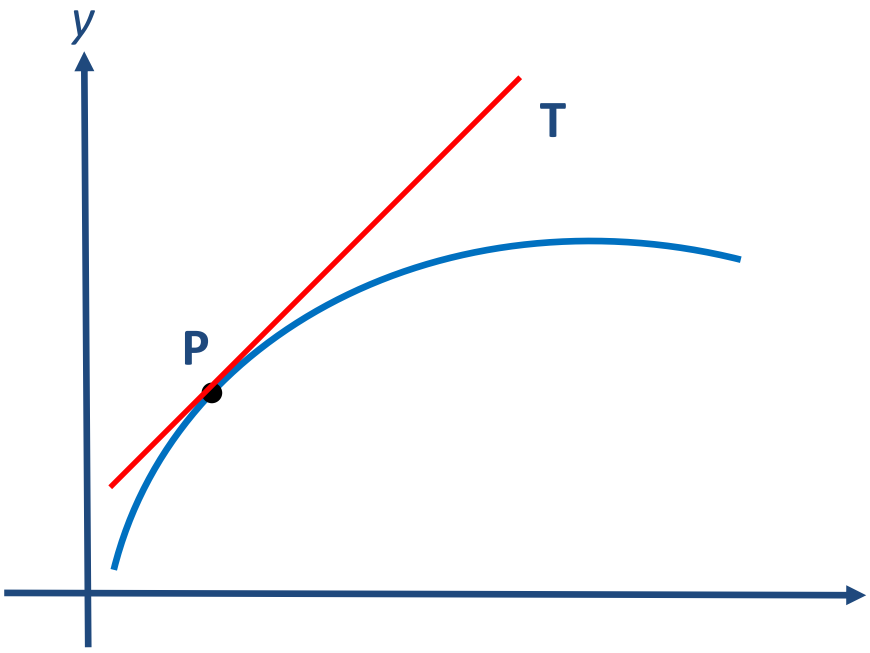
Applications of Differentiation

1. Curve/ Slope
2. Related Rates
3. Max & Min Points

Curve Slope at a Point



The gradient curve at a point can be defined as the gradient of tangent line at the point of the curve. Assume that PT is a tangent to the curve at P. Let P be a point moving along the curve AB. When P is moving from A to B, the tangent to the curve is also moving. In other words, the gradient of the tangent to the curve is changing and it can be obtained by substituting the coordinate points in or .



Example

Find the slope for the curve $y = x^2 + 3x - 4$ at point (3,14)

Differentiate y with respect to x

$$\frac{dy}{dx} = 2x + 3$$

When $x = 3$

$$\frac{dy}{dx} = 2(3) + 3 = 9$$

Thus the slope for the curve $y = x^2 + 3x - 4$ at point is 9 .

Example

Find the slope for the curve $y = x + \dfrac{4}{x}$ at point (1,5)

Differentiate y with respect to x

$$\frac{dy}{dx} = 1 - \frac{4}{x^2}$$

When $x = 1$ at

$$\frac{dy}{dx} = 1 - \frac{4}{1^2} = -3$$

The slope for the curve is -3. Negative sign indicates the slope is decreasing when $x = 1$.

Example

Find the slope for the curve $y = \dfrac{4x^2 + x - 2}{2x - 1}$ at $x = 2$.

Using quotient rule,

$$u = 4x^2 + x - 2 \qquad v = 2x - 1$$

$$u' = 8x + 1 \qquad v' = 2$$

$$\frac{dy}{dx} = \frac{8x^2 - 8x - 3}{(2x-1)^2}$$

When $x = 2$

$$\frac{dy}{dx} = \frac{8(4) - 16 - 3}{3^2} = \frac{13}{9}$$

Thus the slope for the curve $y = \dfrac{4x^2 + x - 2}{2x - 1}$ at point is $\dfrac{13}{9}$.

Example

Find the equation of tangent line for the curve $x^2 - 4xy + y^2 = 3$ at (-2,1)
.

Differentiate using implicit differentiation, we obtain

$$(2y-4x)\frac{dy}{dx} = 4y-2x$$

Slope of the tangent at point $(-2,1)$

$$\left.\frac{dy}{dx}\right|_{(2,-1)} = \frac{4(1)-2(-2)}{\left(2(1)-4(-2)\right)} = \frac{4}{5}$$

Equation of tangent line at (-2,1) is given by

$$y-1 = \frac{4}{5}(x+2)$$

$$5y = 4x+13$$

Example

The parametric equations of a curve given

$$x = t - \frac{1}{t}, \quad y = t + \frac{1}{t} \qquad \text{where} \qquad t \neq 0$$

Find the coordinates of the points on the curve where its gradient is zero and also find the tangent equation at $t = 2.$

Differentiate the parametric equation

$$\frac{dx}{dt} = 1 + \frac{1}{t^2}, \quad \frac{dy}{dt} = 1 - \frac{1}{t^2}$$

Slope of the tangent at point is zero, thus

$$\frac{dy}{dx} = \frac{t^2 - 1}{t^2 + 1} = 0 \qquad t = \pm 1$$

The coordinates of the points are (0,2) and (0,-2)

$$t=1:\ x=1-\frac{1}{1}=0,\quad y=1+\frac{1}{1}=2,$$
$$t=-1:\ x=-1-\frac{1}{-1}=0,\quad y=-1+\frac{1}{(-1)}=-2$$

when $t=2$

$$t=2:\ x=2-\frac{1}{2}=\frac{3}{2},\quad y=2+\frac{1}{2}=\frac{5}{2},$$
$$\left.\frac{dy}{dx}\right|_{t=2}=\frac{2^2-1}{2^2+1}=\frac{3}{5}$$

Equation of tangent is given by

$$y-\frac{5}{2}=\frac{3}{5}(x-\frac{3}{2})$$
$$5y=3x+8$$

Rates of Change

- If is a function of , then is the change of changes with respect to . When represents time, then we usually use the symbol . In which case is a measure of the rate of change in . For example, if represents the radius of a circle in meters and represents time in seconds, then represents the rate of change of radius.

- The chain rule, is sometime used in solving problems involving rates of change. Other than that, implicit differentiation can also be applied to solve related rates and will gives fastest results.

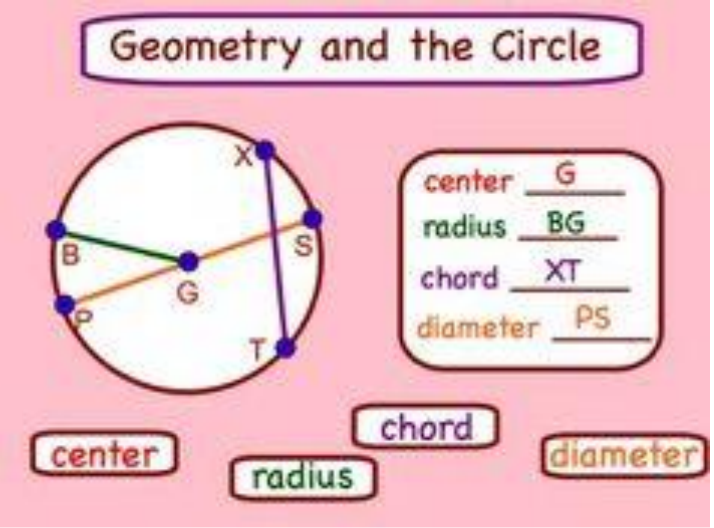
Procedure for solving rate of change problems:

1. List all the given quantities, rewrite problems using mathematical notations
2. Use equation that relate with the changing quantities. Take derivative to get the relating rates
3. Solve by using chain rule or implicit differentiation.

Example

Circle

The radius of a circle is increasing at a rate of 10 cm/s. How fast is the area is increasing at the instant when the radius has reached 5 cm? [area of circle = πr^2]



Given , $\frac{dr}{dt}=10\,\text{cm/s}$

Find $\frac{dA}{dt}$ when $r=5$ cm

1. List and rewrite mathematical notation

From $A=\pi r^2$, differentiate with respect to r and substitute value when $r=5$ cm

2. Use related equations & take derivatives

$$\frac{dA}{dr}=2\pi r=2\pi(5)=10\pi$$

Applying chain rule,

$$\frac{dA}{dt}=\frac{dA}{dr}\times\frac{dr}{dt}$$

$$=10\pi\,\text{cm}\times 10\text{ cm/s}$$

$$=314\text{ cm}^2\text{/s}$$

3. Solve using chain rule

The area of circle increase at rate $314\text{ cm}^2/\text{s}$.

Example

Spherical

A spherical balloon is inflated with air at the rate of $200cm^3/s$. Find the rate of change of the radius of the balloon when the radius is 6cm. Volume of sphere = $4/3\pi r^3$]

Given , $\dfrac{dV}{dt} = 200\,\text{cm}^3/\text{s}$

Find $\dfrac{dr}{dt}$ when $r = 6$ cm

Method 1: Chain Rule

From $V = \dfrac{4}{3}\pi r^3$, differentiate with respect to r and substitute value when $r = 6$ cm

$$\frac{dV}{dr} = 4\pi r^2 = 4\pi(6)^2 = 144\pi$$

Applying chain rule,

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$200 = 144\pi \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = 0.4421$$

Method 2 : Implicit

From $V = \dfrac{4}{3}\pi r^3$, differentiate by using implicit differentiation

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$= 4\pi r^2 \frac{dr}{dt}$$

Substitute value when $r = 6$ cm

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

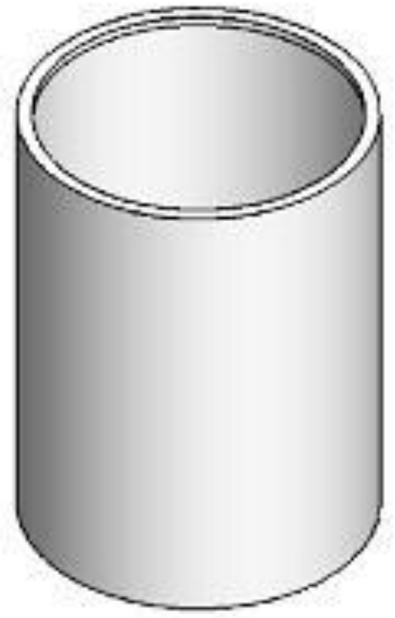
$$200 = 4\pi(6)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = 0.4421$$

Example

Cylindrical

A cylindrical tank has a base with radius, r = 2m. The tank is filled with water at the rate of $2m^3$/min. Find the rate of change of the water level in the tank. [Volume of a cylinder = $\pi r^2 h$]



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Example

Cone

A tank of water in the shape of a cone is leaking water at a constant rate of $2m^3$/hour. The base radius of the tank is 5m and the height of the tank is 14m. [Volume of a cone = $1/3\ \pi r^2 h$]

(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6m?

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6m?

Example

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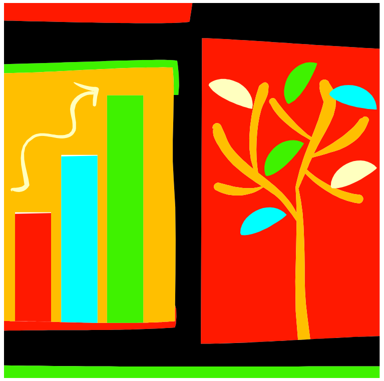
Motion

A body moves along a straight line according to the law,

$$s = \frac{1}{2}t^3 - 2t.$$

Determine its velocity and acceleration after $2s$.



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Example

When a ball is thrown straight-up in the air, its position may be measured as the vertical distance from the ground. Regards 'up' as the positive direction, and let $s(t)$ be the height of the ball in feet after t second, suppose that

$$s(t) = -16t^2 + 128t + 5$$

(a) What is the velocity after 2s

(b) What is the acceleration after 2s?

(c) At what time is the velocity -32m/s?

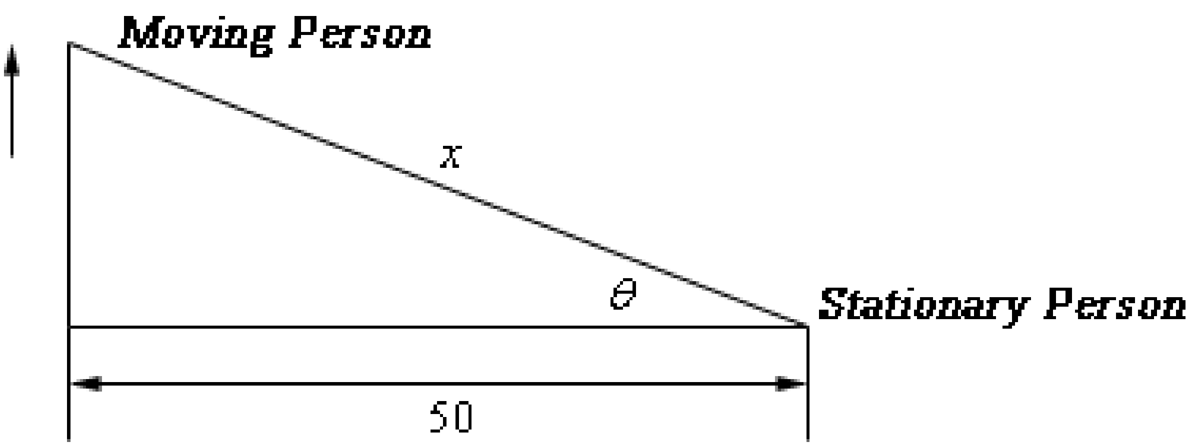
(d) When is the ball at height of 117m?

Example

A 15m ladder is resting against the wall. The bottom is initially 10m away from the wall and is being pushed towards the wall at a rate of ¼ m/s. How fast is the top of the ladder moving up the wall if the bottom of the ladder is 7m away from the wall?

Example

Two people are 50m apart. One of them starts walking north at a rate so that the angle shown in the diagram below is changing at a constant rate of 0.01rad/min. At what rate is distance between the two people changing when θ = 0.5rad?



Conclusion

- In integration by substitution, making appropriate choices for *u* will come with experience.
- Selecting u for by part techniques should follow the LATE guideline.
- If the power of denominator is less than the power of numerator, then the fraction is called proper fraction

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