Exercise 4: Differentiation

Topic 4.1 : Parametric Differentiation

1. Find $\frac{dy}{dx}$ in terms of t for the following parametric equations

(a) $x = 3t^2 - 3t^3$ and $y = t(t^2+3)$	(b) $x = \frac{t}{t-1}$ and $y = t^2 - 1$
(c) $x = 3\sin t$ and $y = e^t$	(d) $x = 5\cos^3 t$ and $y = 7\sin^3 t$

[(a) $\frac{t^2+1}{t(2-3t)}$ (b) $-2t(t-1)^2$ (c) $\frac{e^t}{3\cos t}$ (d) $-\frac{7}{5}\tan t$]

2. Given

$x = 2t$ and $y = 4 - 4t - 4t^2$

(a) Find $\frac{dy}{dx}$ by using parametric differentiation

(b) y in term of x and hence find $\frac{dy}{dx}$

[(a) $-4t-2$, (b) $-2-2x$]

3. Find the $\frac{dy}{dx}$ for the following functions

(a)	$x = t^2 - 2t$	$y = t^3 - 3t$	$t = 4$
(b)	$x = \frac{3t}{1+t^3}$	$y = \frac{3t^2}{1+t^3}$	$t = 2$

[(a) $\frac{15}{2}$, (b) $\frac{4}{5}$]

4. A curve has the following parametric equation

$$x = t - \frac{1}{t} \text{ and } y = t + \frac{1}{t} \qquad t \neq 0$$

Find the coordinates of the point when $\frac{dy}{dx} = 0$

[$(0, 2)$ $(0, -2)$]

5. Find $\frac{dy}{dx}$ when

$$x = \cos 2\theta \text{ and } y = 2\theta + \sin 2\theta$$

[$-\cot\theta$]

Topic 4.2 : Higher Derivatives

6. Find the first and second derivatives of the following function

(a) $f(x) = x^5 + 6x^2 - 7x$ (b) $y = x^2 \cos x$ ✓ (c) $y = \sqrt{3x^2 + 1}$

(d) $f(x) = \frac{x+3}{x^2 - 2x}$ ✓ (e) $y = \frac{x}{1-x}$ (f) $\sqrt{x} + \sqrt{y} = 1$ ✓

(g) $y = \sqrt{x} + \sqrt[3]{x}$ (h) $x^2 + 3xy = \sqrt[3]{y}$

[Ans: (a) $f''(x) = 20x^3 + 12$ (b) $y'' = (2 - x^2)\cos x - 4x \sin x$ (c) $y'' = \frac{3}{(3x^2+1)^{\frac{3}{2}}}$ (d) $y'' = \frac{2x^3 + 18x^2 - 36x + 24}{(x^2 - 2x)^3}$ (e) $y'' = \frac{2}{(1-x)^3}$ (f) $y'' = \frac{1}{2}x^{-\frac{3}{2}}$ (g) $y'' = \frac{-1}{4\sqrt{x^3}} - \frac{2}{9\sqrt[3]{x^5}}$ (h) $y'' = \frac{3(2x+3y)\sqrt[3]{y^2}}{1 - 9x\sqrt[3]{y^2}}$]

7. Find $\frac{d^3y}{dx^3}$ for $y = 2x^5 + 3x^3 - 4x + 1$

[$120x^2 + 18$]

Topic 4.3 : Implicit Differentiation

8. Find the $\frac{dy}{dx}$ by implicit differentiations

(a) $\frac{1}{x} + \frac{1}{y} = 1$ (b) $3y^2 - 2x^2 = 2xy$ ✓ (c) $y = x\sin^2 y + 3xy$

(c) $xy = 25$ (e) $x^2 + 3xy + y^2 = 15$ ✓ (f) $(x+y)^3 + 3y = 3$

(g) $x^3 - y^3 = 1$ (h) $xy + y^2 = 1$ at the point (0,-1)

[Ans: (a) $\frac{y-1}{1-x}$ (b) $\frac{4x+2y}{6y-2x}$ (c) $\frac{\sin^2 y+3y}{1-2x\sin y\cos y-3x}$ (d) $-\frac{y}{x}$ (e) $\frac{-2x-3y}{3x+2y}$ (f) $\frac{-(x+y)^2}{(x+y)^2+1}$ (g) $\frac{x^2}{y^2}$ (h) $-\frac{1}{2}$]

9. Find the $\frac{dy}{dx}$ by implicit differentiations

(a) $xy+2y=3$ (b) $2x^2-xy+2y=5$ (c) $x^{1/2}+y^{2/3}=x$

(d) $xe^y+ye^x=2x$ (e) $xy=\sin(x+y)$ (f) $\ln\left(\frac{x^2}{y}\right)=x$ ✓

(g) $e^{xy}+\ln y^2=x$ ✓ (h) $2xy+3y=x^2$ (i) $x+\frac{1}{y}=5$

(i) $x^2+y^2=25$ (k) $x^2+y=x^3+y^2$ (l) $x^3-y^3=xy$

(m) $x^2y+e^{-2x}y^2=2x$ (n) $\sin(x+y)^2=y$ ✓ (o) $\sqrt{2x}+y^2=4$

(p) $(x-2y)^2=y$ (q) $(2x+y)^3=x$ (r) $xy-\sqrt{x}=y+2$

[Ans: (a) $\frac{-y}{x+2}$ (b) $\frac{y-4x}{2-x}$ (c) $\frac{3(2x^{1/2}-1)y^{1/3}}{4x^{1/2}}$ (d) $\frac{2-e^y-ye^x}{xe^y+e^x}$ (e) $\frac{\cos(x+y)-y}{x-\cos(x+y)}$ (f) $\frac{y(2-x)}{x}$ (g) $\frac{y(1-ye^{xy})}{xye^{xy}+2}$ (h) $\frac{2(x-y)}{2x+3}$ (i) y^2 (j) $\frac{-x}{y}$ (k) $\frac{x(3x-2)}{1-2y}$ (l) $\frac{3x^2-y}{x+3y^2}$ (m) $\frac{2(1-xy+e^{-2x}y^2)}{x^2+2ye^{-2x}}$ (n) $\frac{2(x+y)\cos(x+y)^2}{1-2(x+y)\cos(x+y)^2}$ (o) $\frac{-1}{2y\sqrt{2x}}$ (p) $\frac{2(x-2y)}{1+4(x-2y)}$ (q) $\frac{1}{3(2x+y)^2}-2$ (r) $\frac{2y\sqrt{x}-1}{2\sqrt{x}(1-x)}$]

10. If $xy+y^2=1$, use implicit differentiation to find $\frac{d^2y}{dx^2}$ at the point $(0,1)$

[$-\frac{1}{4}$]