Calculus
Limit & Continuity

By

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Description

Aims

This chapter is aimed to :

1. introduce limit at infinity
2. explain the continuity of the function
3. check the continuity using continuity test.

Expected Outcomes

1. Students should be able to explain limit at infinity.
2. Students should be know the continuity by looking at the graph.
3. Studenst should be able to determine continuity at a given point by referring to continuity test.

References

1. Abdul Wahid Md Raji, Hamisan Rahmat, Ismail Kamis, Mohd Nor Mohamad, Ong Chee Tiong. ***The First Course of Calculus for Science & Engineering Students***, Second Edition.

Content

1. Limit at Infinity
2. Continuity

Limit at a Infinity

We are interested in examining the limiting behavior of functions as x increases without bound (written $x \to \infty$) or as x decreases without bound (written $x \to -\infty$).

$$\boxed{\lim_{x\to\infty} k = k}$$

$$\boxed{\lim_{x\to\infty} \frac{1}{x^n} = 0}$$

Example:

$$\lim_{x\to+\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to+\infty} 5 + \lim_{x\to+\infty}\frac{1}{x} = 5+0=5$$

Note: To evaluate the limit at a infinity of rational function, divide the numerator and denominator by the largest power of the variable that appear in the denominator

Example

Evaluate $\lim\limits_{t\to\infty}\dfrac{12t^2-15t+12}{t^2+1}$

Highest power in denominator is 2, so we can divide our fraction with t^2

$$\lim_{t\to\infty}\frac{\left(12t^2-15t+12\right)/t^2}{\left(t^2+1\right)/t^2}=\lim_{t\to\infty}\frac{12-\frac{15}{t}+\frac{12}{t^2}}{1+\frac{1}{t^2}}=\frac{\lim\limits_{t\to\infty}\left(12-\frac{15}{t}+\frac{12}{t^2}\right)}{\lim\limits_{t\to\infty}\left(1+\frac{1}{t^2}\right)}$$

$$=\frac{\lim\limits_{t\to\infty}12-\lim\limits_{t\to\infty}\frac{15}{t}+\lim\limits_{t\to\infty}\frac{12}{t^2}}{\lim\limits_{t\to\infty}1+\lim\limits_{t\to\infty}\frac{1}{t^2}}$$

$$=\frac{12-0+0}{1+0}$$

$$=12$$

Example

Evaluate the following limit.

(a) $\lim\limits_{x\to\infty} \dfrac{x^2+x-6}{x-2}$ (b) $\lim\limits_{t\to\infty} \dfrac{\sqrt{t}-1}{t-1}$ (c) $\lim\limits_{x\to\infty} \dfrac{2x^3-2}{x^3+1}$

(d) $\lim\limits_{x\to\infty} e^{\frac{1}{x}}$ (e) $\lim\limits_{x\to\infty} \tan^{-1} x$

(a) Highest order in denominator is 1, so divide fraction by x

$$\lim_{x\to\infty} \frac{x^2+x-6}{x-2} = \lim_{x\to\infty} \frac{\left(x^2+x-6\right)/x}{\left(x-2\right)/x}$$

$$= \lim_{x\to\infty} \frac{x+1-\frac{6}{x}}{1-\frac{2}{x}}$$

$$= \infty$$

Example

(b) Highest order in denominator is 1, so divide fraction by t

$$\lim_{t\to\infty}\frac{\sqrt{t}-3}{t-1}=\lim_{t\to\infty}\frac{\left(\sqrt{t}-1\right)/t}{\left(t-1\right)/t}$$

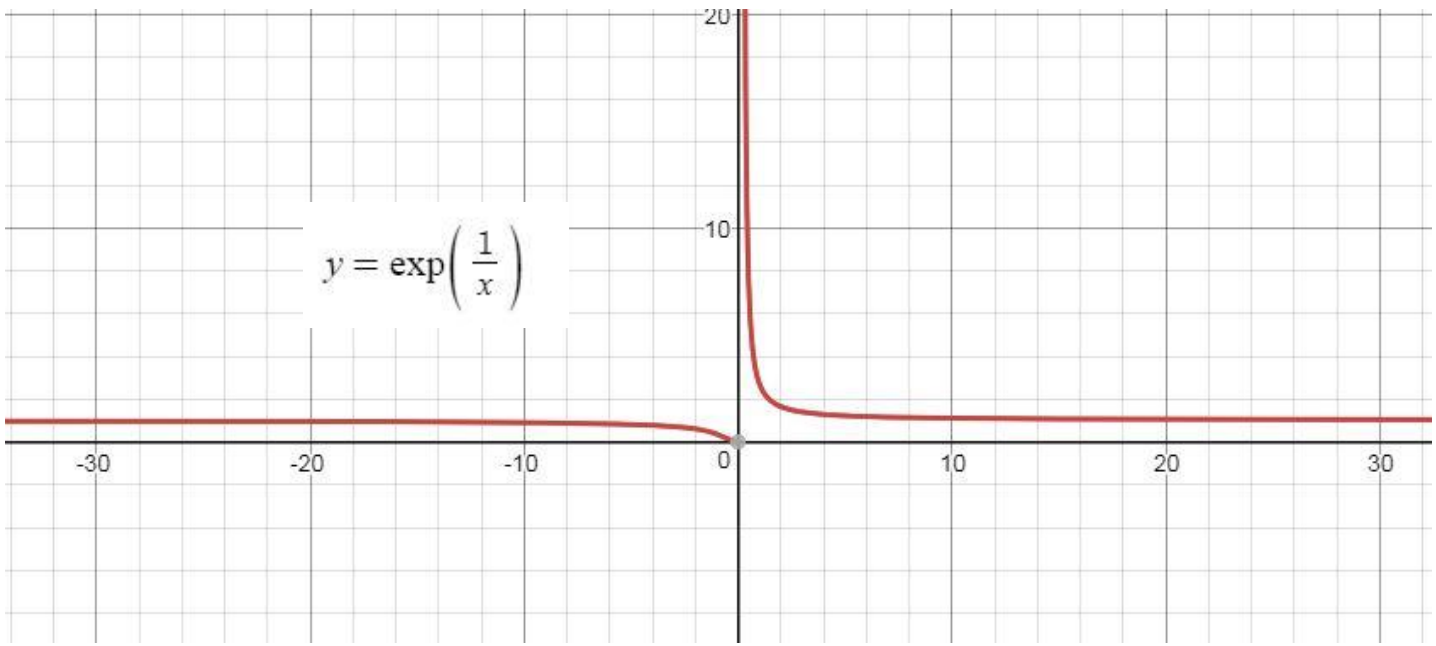
$$=\lim_{t\to\infty}\frac{\frac{1}{\sqrt{t}}-\frac{3}{t}}{1-\frac{1}{t}}=0$$

(c) The denominator has power 3, therefore

$$\lim_{x\to\infty}\frac{2x^3-2}{x^3+1}=\lim_{x\to\infty}\frac{\left(2x^3-2\right)/x^3}{\left(x^3+1\right)/x^3}$$

$$=\lim_{x\to\infty}\frac{2-\frac{2}{x^3}}{1+\frac{1}{x^3}}=2$$

(d) $e^{\frac{1}{x}}$ not a fractional function, with the aid of mathematical software we evaluate the limit of the function graphically.



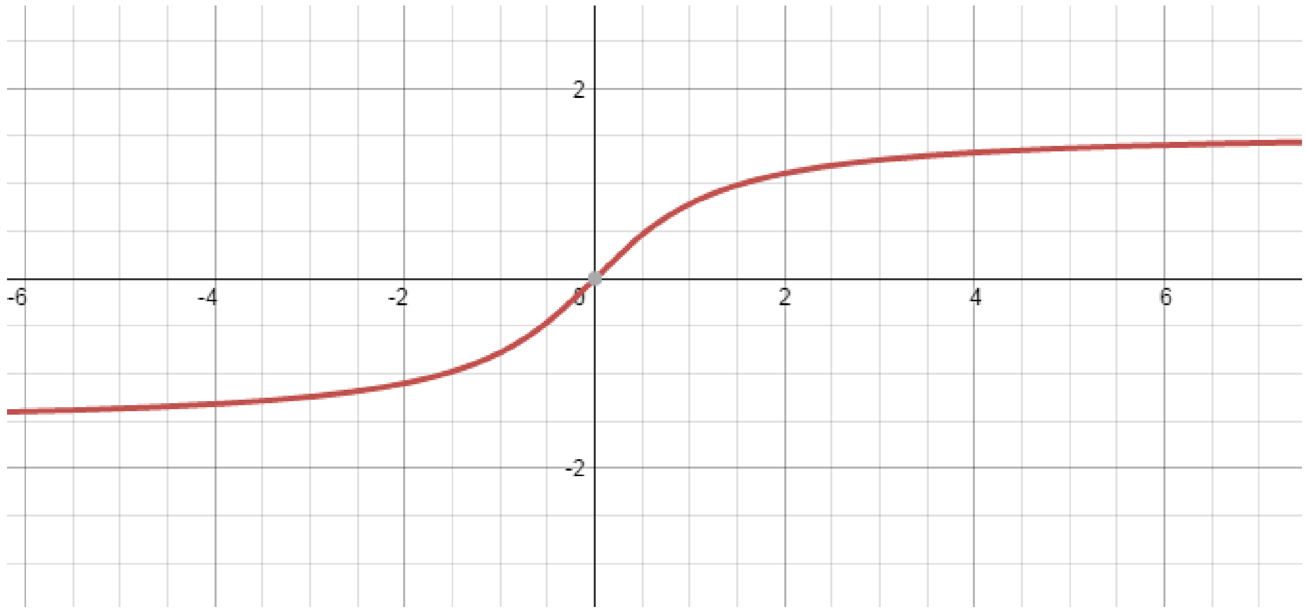
$$\lim_{x \to -\infty} f(x)$$

$$\lim_{x \to +\infty} e^{1/x} = 1$$

$$\lim_{x \to +\infty} f(x)$$

https://www.desmos.com/calculator

(d) $\tan^{-1} x$ is an inverse trigonometric function, with the aid of mathematical software we evaluate the limit of the function graphically.



$$\lim_{x \to -\infty} f(x)$$

$$\lim_{x \to +\infty} \tan^{-1} x$$

$$\lim_{x \to +\infty} f(x)$$

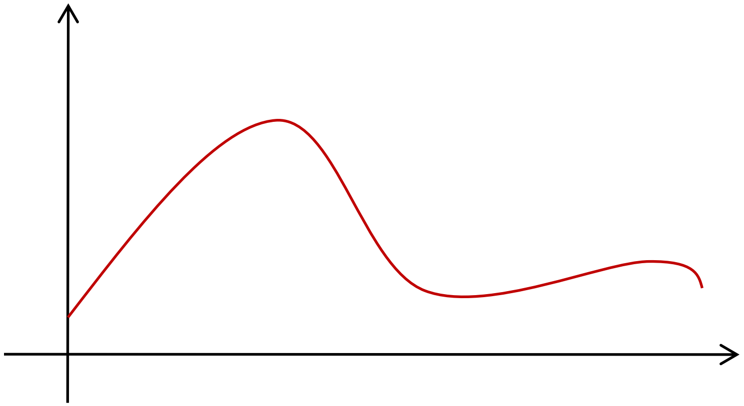
https://www.desmos.com/calculator

Continuity

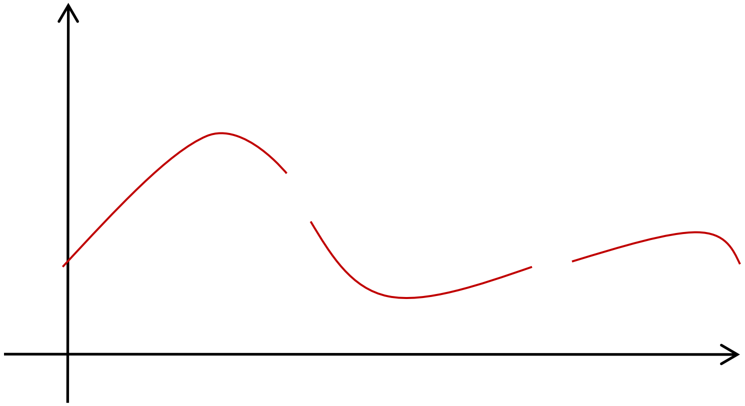
- A function can be divided into two categories : continuous or discontinuous.
- Continuous function flows nicely in smooth direction without interruption. Opposite pattern are observed for discontinuous functions.
- One of the easy way to detect the continuity of the function is by looking at the graph : if its graph can be drawn without lifting a pen/pencil, then the graph is said to be continuous.
- We will discuss the continuity of a graph (of a function) in the next slides.

❑ Continuity of a function:

CONTINUOUS:

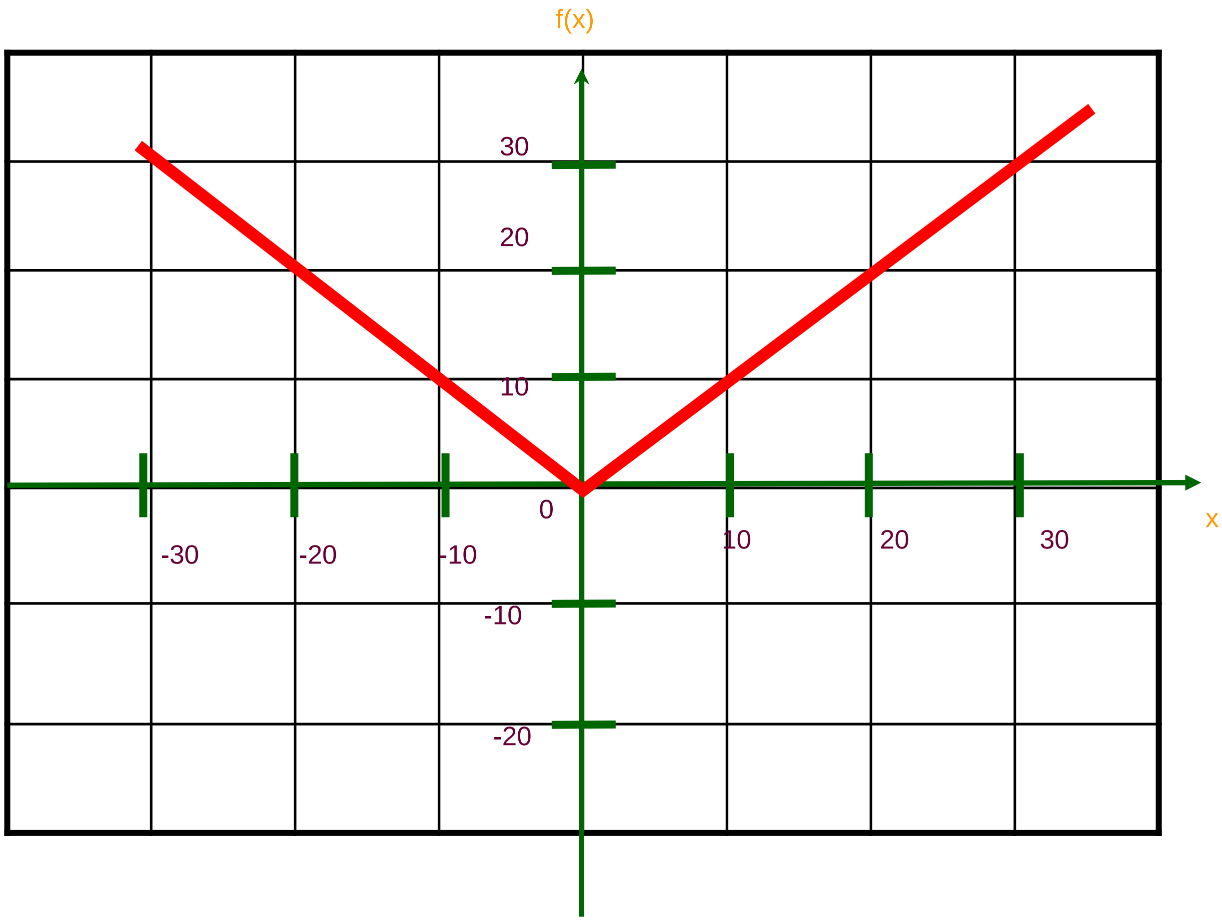


NOT CONTINUOUS:



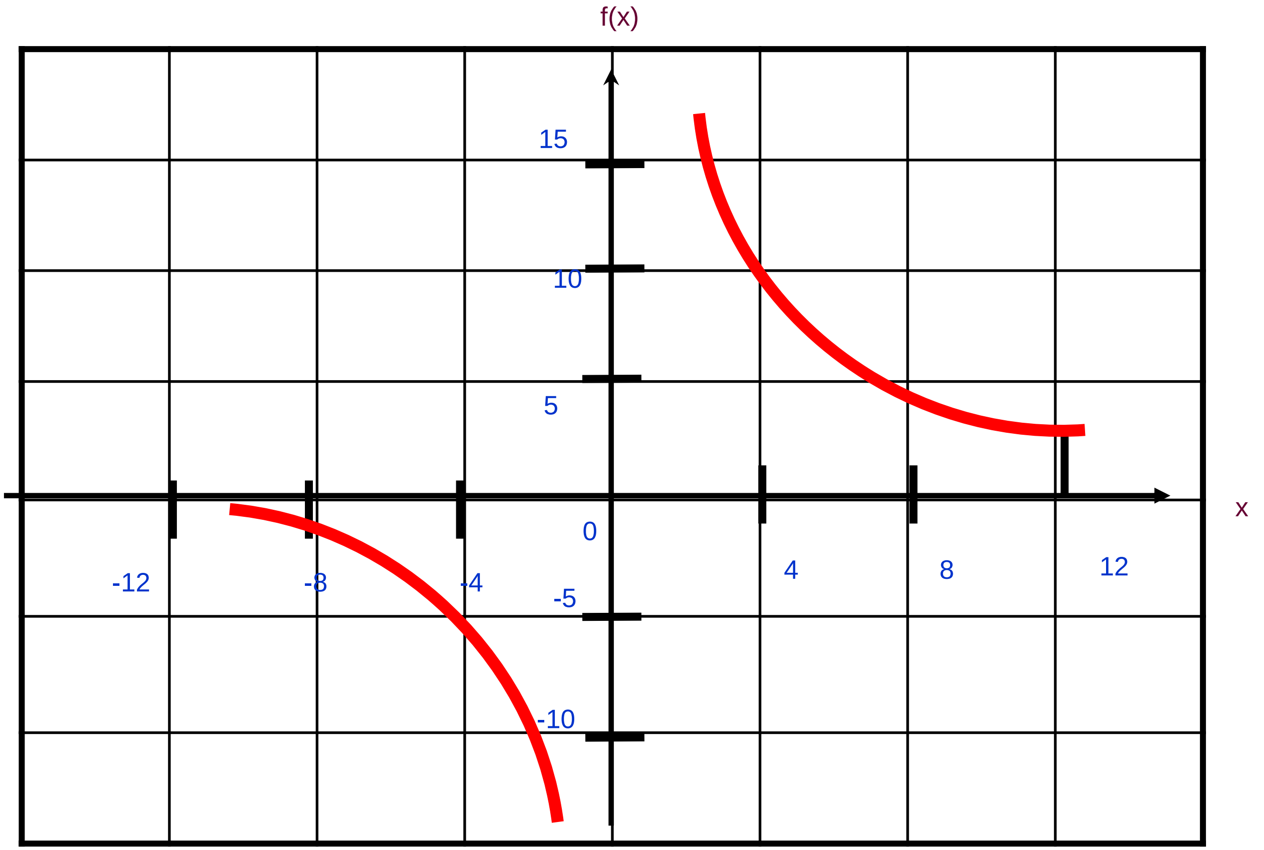
Example

Determine the continuity of the following graph.



Example

Determine the continuity of the following graph.



The Continuity Test

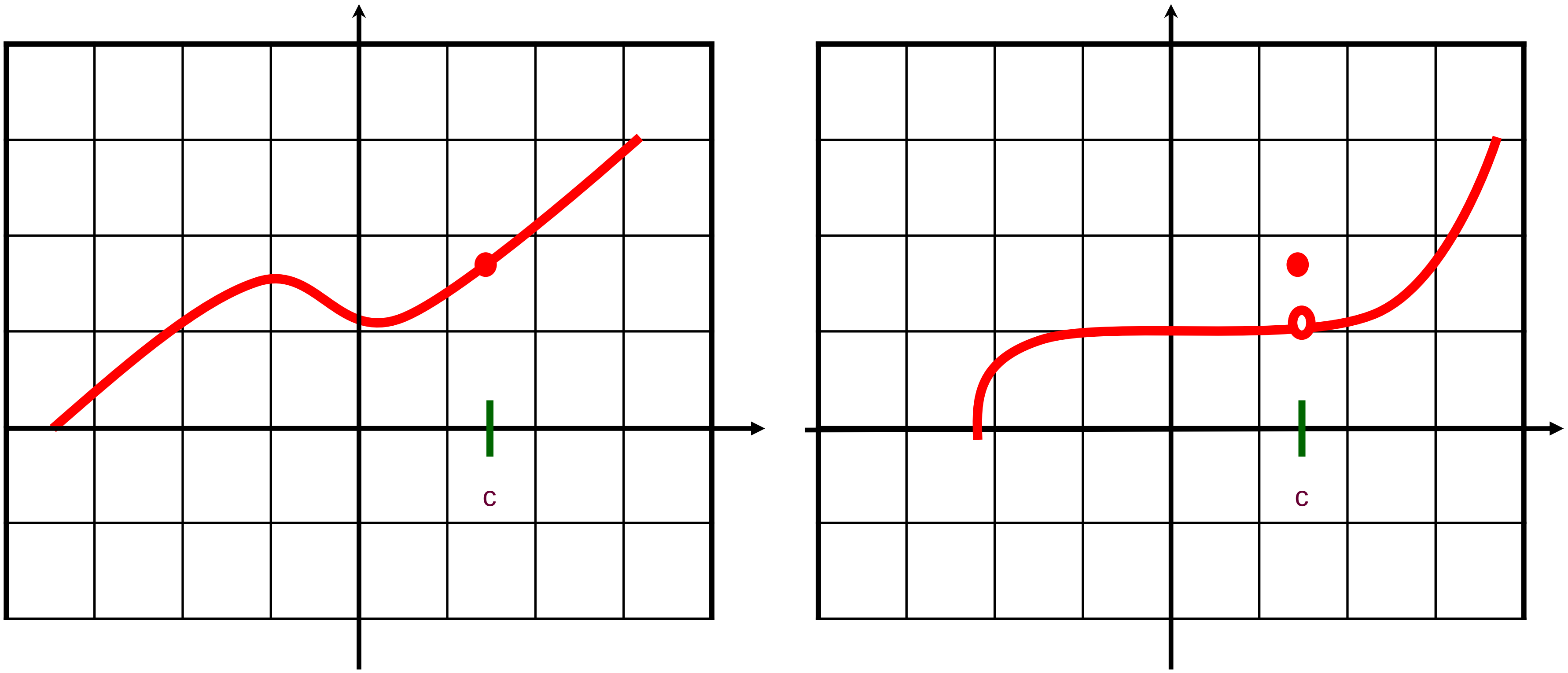
Definition – Continuity: A function is said to be continuous at a point $x = c$ if the following conditions are satisfied.

1. The function f is defined at $x = c$, that is $f(c)$ exist.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

Failure to meet one of these conditions, then the function is discontinuous at $x = c$.

Example

Determine the continuity of the following graphs.



$\lim_{x\to c} f(x) = f(c)$

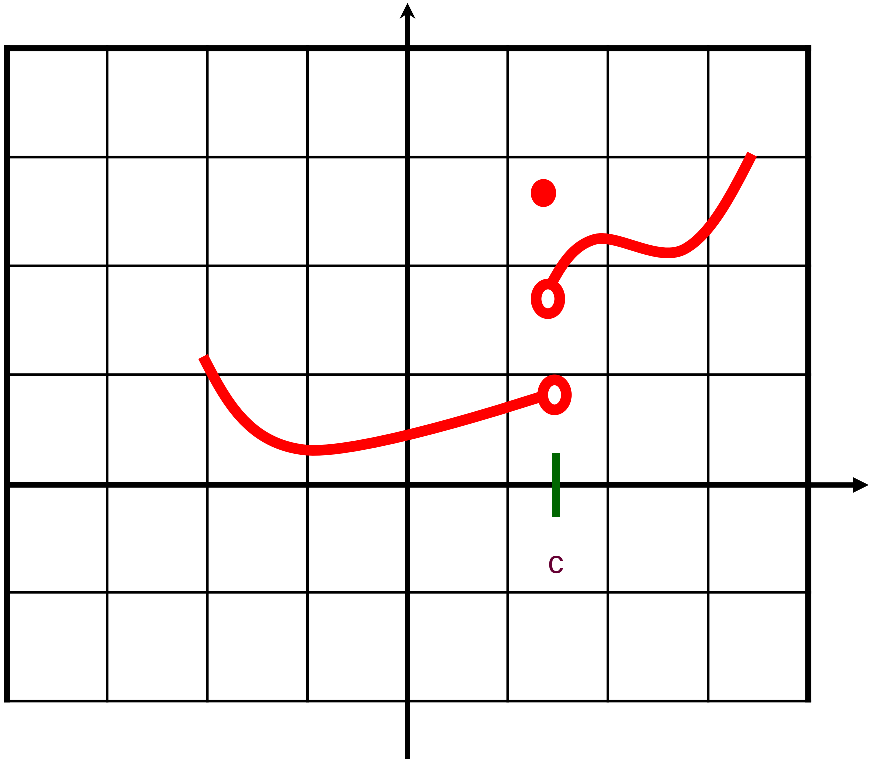
Hence, $f(x)$ **continuous at** c **.**

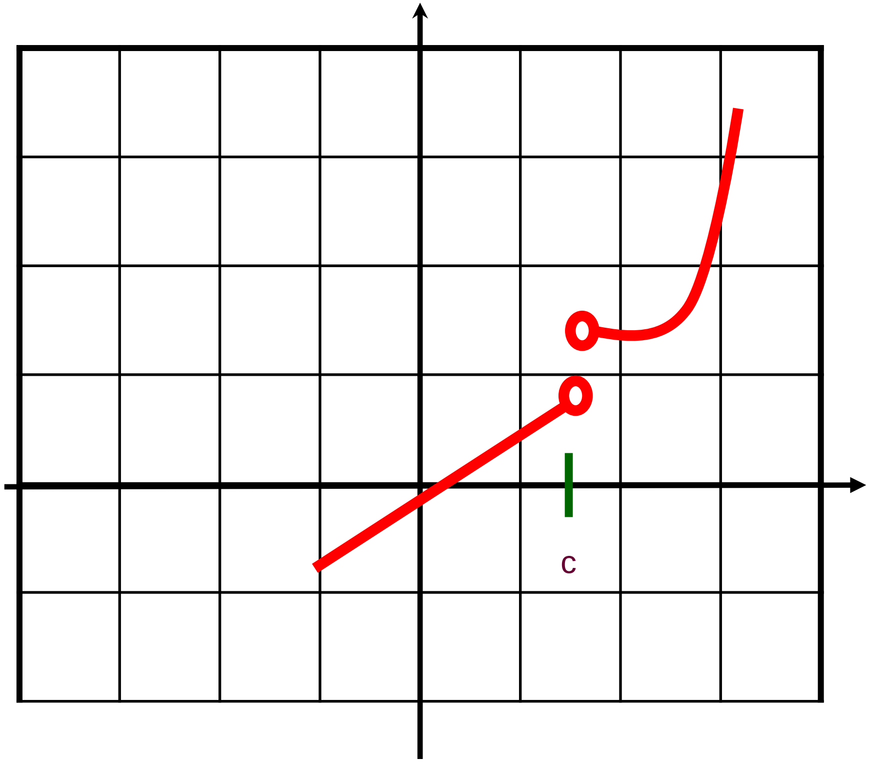
$\lim_{x\to c} f(x)$ **exist but** $\lim_{x\to c} f(x) \neq f(c)$

Hence, $f(x)$ **discontinuous at** c **.**

Example

Determine the continuity of the following graphs.





$f(c)$ **exist but** $\lim\limits_{x\to c} f(x)$ **does not exist**

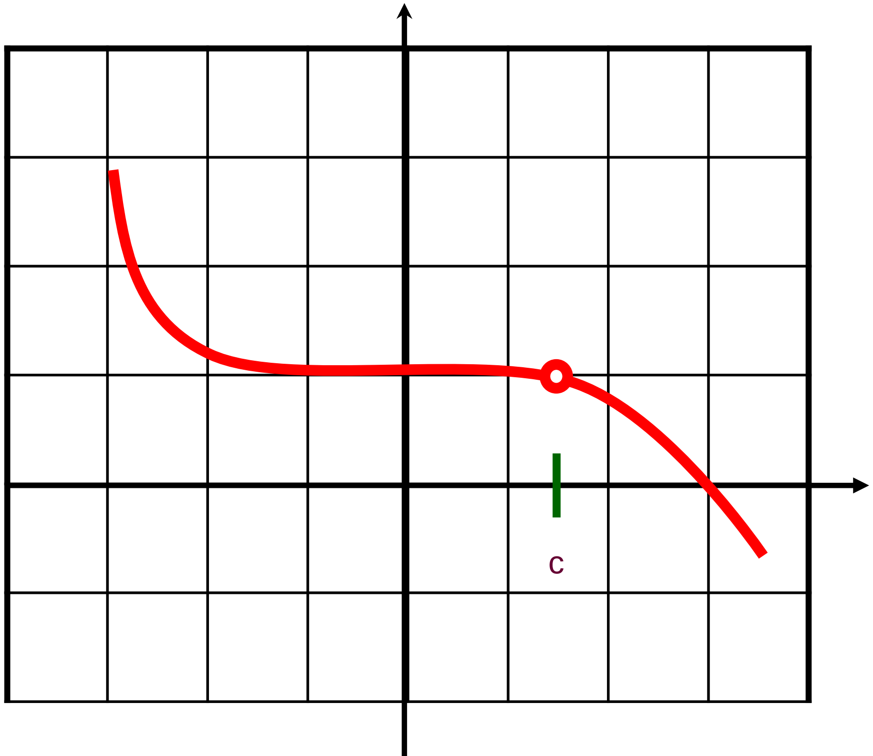
Hence, $f(x)$ **discontinuous at** c **.**

$\lim\limits_{x\to c} f(x)$ **and** $f(c)$ **does not exist.**

Hence, $f(x)$ **not continuous at** c **.**

Example

Determine the continuity of the following graphs.



$\lim_{x \to c} f(x)$ **exist but** $f(c)$ **does not exist**

Hence, $f(x)$ **not continuous at** c **.**

Example

Determine whether the function $f(x)=\dfrac{x^2-x-2}{x-2},\ x=1$ is continuous at given x value?

By using the continuity test,

1 Function is defined at $x=1$.

$$f(c)=f(1)=\frac{x^2-x-2}{x-2}=\frac{1-1-2}{1-2}=2$$

2 Limit is exist

$$\lim_{x\to1} f(x)=\lim_{x\to1}\frac{x^2-x-2}{x-2}=\frac{\lim\limits_{x\to1}x^2-\lim\limits_{x\to1}x-\lim\limits_{x\to1}2}{\lim\limits_{x\to1}x-\lim\limits_{x\to1}2}=2$$

3 $\lim\limits_{x\to1} f(x)=f(1)=2$

Hence, $f(x)=\dfrac{x^2-x-2}{x-2}$ is continuous at $x=1$.

Example

Determine whether the function $f(x)=\frac{x^2-x-2}{x-2}$, $x=2$ is continuous at given x value?

By using the continuity test,

1. Function is **undefined** at $x=2$.

$$f(c)=f(2)=\frac{2^2-2-2}{2-2}=\frac{0}{0}$$

2. Limit is exist

$$\lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{x^2-x-2}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+1)}{x-2}=3$$

indeterminate form

3. $\lim_{x\to 2} f(x)\neq f(2)$

Hence, $f(x)=\frac{x^2-x-2}{x-2}$ is discontinuous at $x=2$.

Example

Discuss the is continuity of piecewise function at $x=0$.

$$f(x)=\begin{cases} x+1, & x<0 \\ 1 & x=0 \\ (\sin x)/x, & x>0 \end{cases}$$

To check the continuity of the graph, we use the continuity test.

- At $x=0$, function is defined $f(0)=1$.

- To check the existence of the limit, we need to evaluate limit from the both sides. Using numerical approach, we obtained:

x approaching 0 from left → ← **x approaching 0 from right**

x	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1
$f(x)$	0.90000	0.9900	0.9990	0.9999	?	0.9999	0.9999	0.9999	0.9983

$$\lim_{x\to 0^-} x+1=1 \qquad \lim_{x\to 0^+}\frac{\sin x}{x}=1$$

Notice that different function are applied on the left and right side. As a conclusion, limit is exist since both sides have the same value.

$$\lim_{x\to 0} f(x)=1$$

- $\lim_{x\to 0} f(x)= f(0)=1$

All requirement are successfully meet. Therefore $f(x)$ is continuous at $x=0$.

Example

Given $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$

Find the value of k so that the function $f(x)$ is continuous

In order for $f(x)$ to be continuous, $f(x)$ has to satisfy all conditions in continuity test. Condition 3 stated that,

$$f(0) = \lim_{x \to 0} f(x)$$

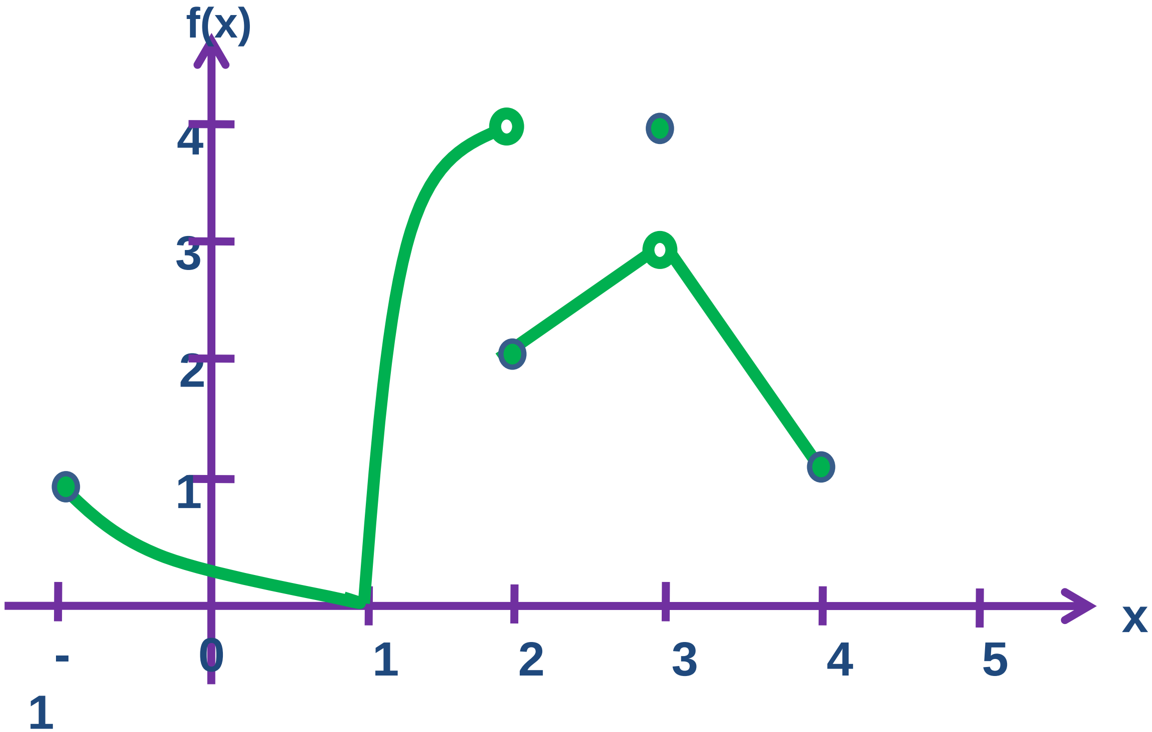
From previous example in slide : $\lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{\sin x}{x} = 1$. On the other hand, is defined at $f(0) = k$. Therefore ,

$$f(0) = \lim_{x \to 0} f(x)$$

$$k = 1$$

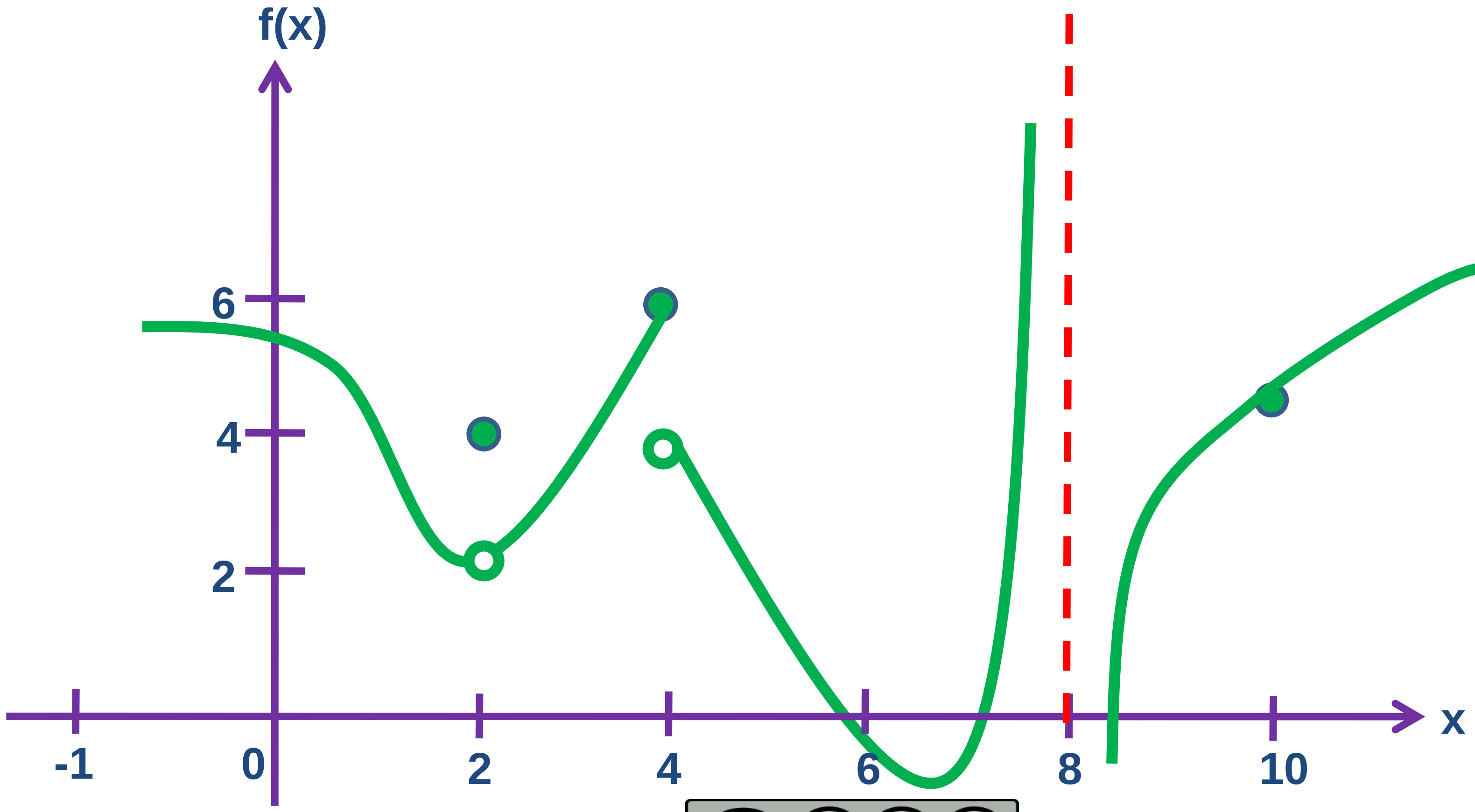
Example

Determine whether the function $f(x)$ is continuous at $x=1, x=2, x=3$ by using continuity test.



Example

Determine whether the function $f(x)$ is continuous at $x=2,\ x=4,\ x=8,\ x=10$ by using continuity test.



Conclusion #1

- Limit at Infinity : $\lim_{x \to +\infty} f(x) = L, \ \lim_{x \to +\infty} k = k, \ \lim_{x \to +\infty} \frac{1}{x^n} = 0$
- In finding limit at infinity for fractional function, division of higher power variable is use by dividing the numerator and denominator of function with , where is the highest power in the denominator
- A function is said to be **continuous** if it graph can be drawn over each interval of its domain with a continuous motion of pen without lifting a pen.

$x = c$

Conclusion #2

- **Continuity Test :** A function is continuous at if and only if it meets the following conditions:

1. $f\left(a\right)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

If any of these three fails, then is discontinuous at $x = c$.

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