Calculus
Limit & Continuity

By

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Description

Aims

This chapter is aimed to :

1. introduce the concept of limit
2. explain the definition of one sided and two sided limit
3. evaluate limit using three different approaches.

Expected Outcomes

1. Students should be able to describe the concept of limits
2. Students should be able explain one sided limit and two sided limit
3. Students should be able to find limit numerically, analytically and graphically

References

1. Abdul Wahid Md Raji, Hamisan Rahmat, Ismail Kamis, Mohd Nor Mohamad, Ong Chee Tiong. ***The First Course of Calculus for Science & Engineering Students***, Second Edition.

Content

1. Limit of a Function
2. Evaluate Limit : Numerical Method
3. Evaluate Limit : Graphical Method

1.1 Limit of a Function

- Limit is the most important concept of all calculus.
- The main ideas of calculus, the derivative and the integral, are defined using limits.
- All you need is to develop an intuitive understanding, and you will see how simple these concepts are.
- The concept of limit study what will happen to a function when variable χ approaches a certain value.

- Consider an example which will help you to understand the concept of limit. Suppose there is a huge forest blaze with a raging fire. Imagine that you are moving closer to the forest, the distance x between you and forest is decreases. As you keep on moving, you start feel heat all over your body. Let the temperature on the surface of your body measured as $f(x)$.

- Now as you getting closer to the fire, increased heat are felt on your body. The closer you get, the greater the sense of heat. Now you would not want to actually put yourself in the fire i.e. $x = 0$, but yet as you get close and close to the fire you have sense that temperature on the surface of your body will increasing until it reaches the temperature of fire.

- In limit, we are not interested in the value of $f(x)$ when $x=0$
- We are more interested in the behaviour of $f(x)$ as x comes closer and closer to a value of c.
- The notation of one sided limit is given as follow

$$\boxed{\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L}$$

Left Side Limit
x **approaches** c **from left.**

Right Side Limit
x **approaches** c **from right.**

1.1 Limit of a Function

When does a limit **EXIST**? A limit exists if and only if both corresponding one sided limits exist and are equal.

Definition – Limit: If the limit from the left and right sides have the same value,

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

Then, $\lim_{x \to c} f(x)$ exist and it is written as

$$\lim_{x \to c} f(x) = L$$

and we read as "the limit $f(x)$ as x approaches c is L"

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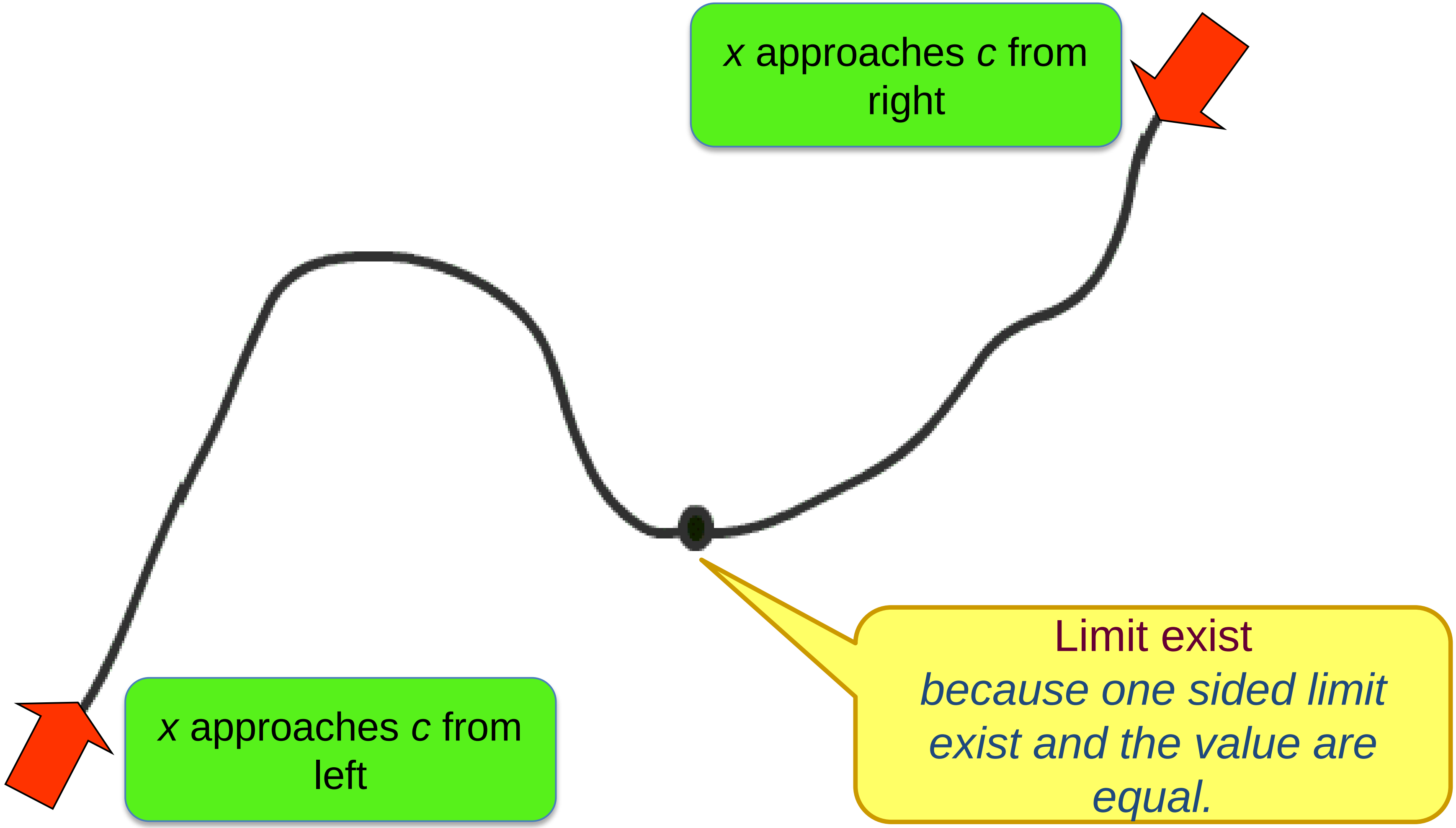
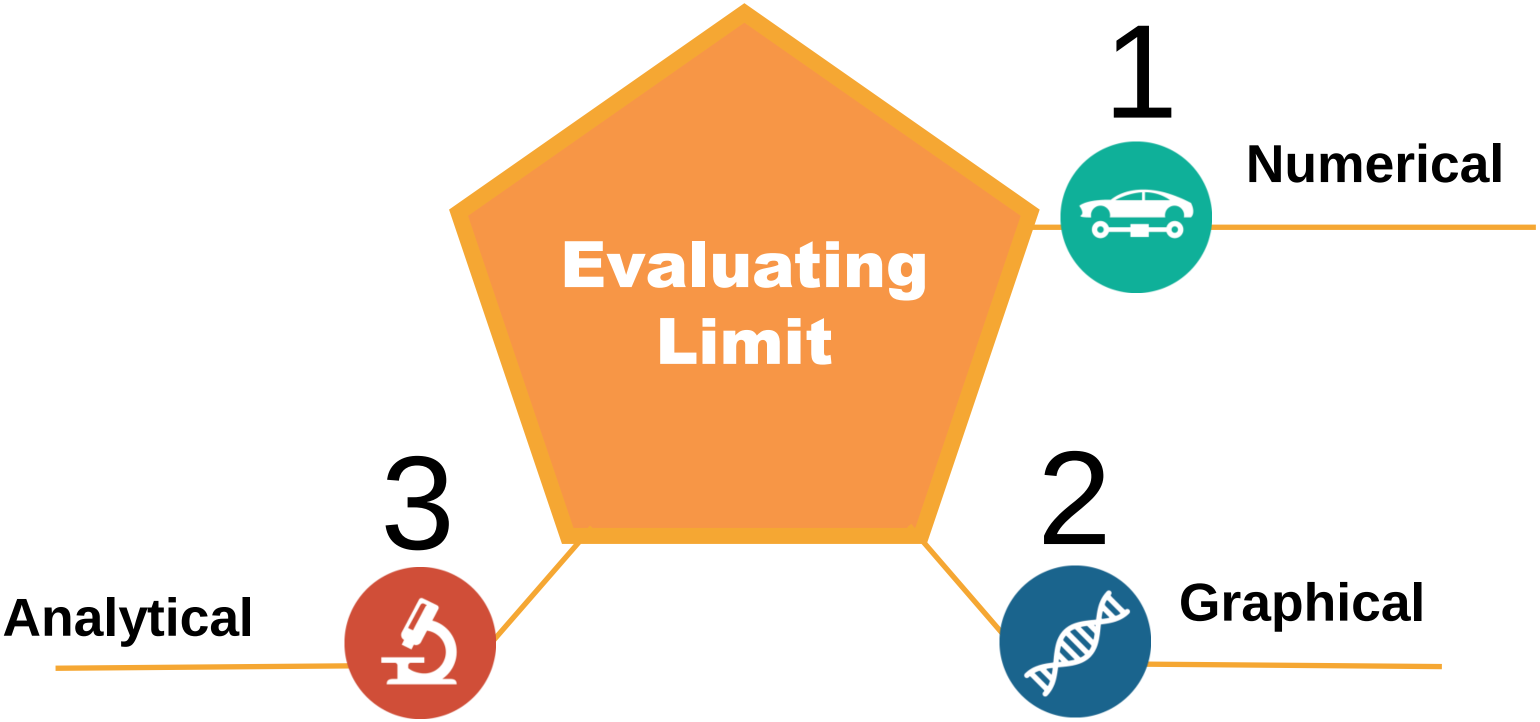

x approaches *c* from right
x approaches *c* from left
Limit exist
because one sided limit exist and the value are equal.

Limit

Limit can be evaluated using three methods.



Numerical Method

In this method, limit is solved by

- inserting an appropriate value of x from left (left side limit) and right (right side limit) and calculate the corresponding $f(x)$.
- By doing so, we are expecting to reach a certain value → LIMIT
- Aim: to be able to interpret limit behavior based on looking at a table of values

Example

Evaluate $\lim_{x \to 1} 2x$ by using table.

Compute limit from both sides as follow

***x* approaching 1 from left** → ← ***x* approaching 1 from right**

x	0.9	0.99	0.999	0.9999	1	1.0001	1.001	1.01	1.1
f(x)	1.8	1.98	1.998	1.9998	?	2.0002	2.002	2.062	2.1

$$\lim_{x \to 1^-} 2x = 2 \qquad \lim_{x \to 1^+} 2x = 2$$

Since limit from left and right (one sided limit) exist and equal, two side limit exist and written as

$$\therefore \lim_{x \to 1} 2x = 2$$

Example

Evaluate $\lim_{x\to 0}\frac{\sin x}{x}$ numerically where x is in radian.

x approaching 0 from left → ← *x* approaching 0 from right

x	-0.1	-0.01	-0.01	-0.0001	0	0.0001	0.001	0.01	0.1
f(x)	0.99833	0.99998	0.99999	0.99999	?	0.99999	0.99999	0.99998	0.99833

$$\lim_{x\to 0^-}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0^+}\frac{\sin x}{x}=1$$

In this example, we shall summarize result as $\therefore \lim_{x\to 0}\frac{\sin x}{x}=1$

Example

Evaluate $\lim_{x\to 2}\frac{3x^2-2x-8}{x-2}$ using numerical method.

***x* approaching 2 from left** → ← ***x* approaching 2 from right**

x	1.9	1.99	1.999	2	2.001	2.01	2.1
f(x)	9.7	9.97	9.997	?	10.003	10.03	10.3

$$\lim_{x\to 2^-} f(x)=10 \qquad \lim_{x\to 2^+} f(x)=10$$

Since the limits from left and right have the same values, then

$$\lim_{x\to 2}\frac{3x^2-2x-8}{x-2}=10$$

Example

Evaluate $\lim_{x\to 0}\frac{1}{x}$ numerically.

x **approaching 0 from left** →

← *x* **approaching 0 from right**

x	-0.1	-0.01	-0.01	-0.0001	0	0.0001	0.001	0.01	0.1
f(x)	-10	-100	-1000	-10000	?	10000	1000	100	10

$$\lim_{x\to 0^-}\frac{1}{x} = -\infty \qquad \lim_{x\to 0^+}\frac{1}{x} = +\infty$$

Both sides have different value, we can concluded that

$$\therefore \lim_{x\to 0}\frac{1}{x} \quad \text{does not exist}$$

Example

Evaluate $\lim_{x\to 0} f(x)$ numerically where $f(x)=\begin{cases} x^2+1, & x<0 \\ x^2-1, & x>0 \end{cases}$.

***x* approaching 0 from left** → ← ***x* approaching 0 from right**

x	-0.1	-0.01	-0.01	0	0.001	0.01	0.1
f(x)	1.01	1.0001	1.000001		-0.999999	-0.999	-0.99

$$\lim_{x\to 0^-}\left(x^2+1\right)=1 \qquad \lim_{x\to 0^+}\left(x^2-1\right)=-1$$

Limit from left and right have different values, then

$$\therefore \lim_{x\to 0} f(x) \text{ does not exist}$$

Graphical Method

In this method, limit is solved through a graph.

From the graph, we can determine the limit exist or not

when does a limit exist?

$$\boxed{\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L}$$

Example

Evaluate $\lim\limits_{x \to 1} 2x$ by graphical method.

Plot graph of $y = 2x$. From the graph plotting in **Figure 1**, as x approached 1 from left, the function f(x) goes to 2. The same thing occur as x approached 1 from right

$$\lim_{x \to 1^-} 2x = \lim_{x \to 1^-} 2x = 2$$

Hence

$$\therefore \lim_{x \to 1} 2x = 2$$

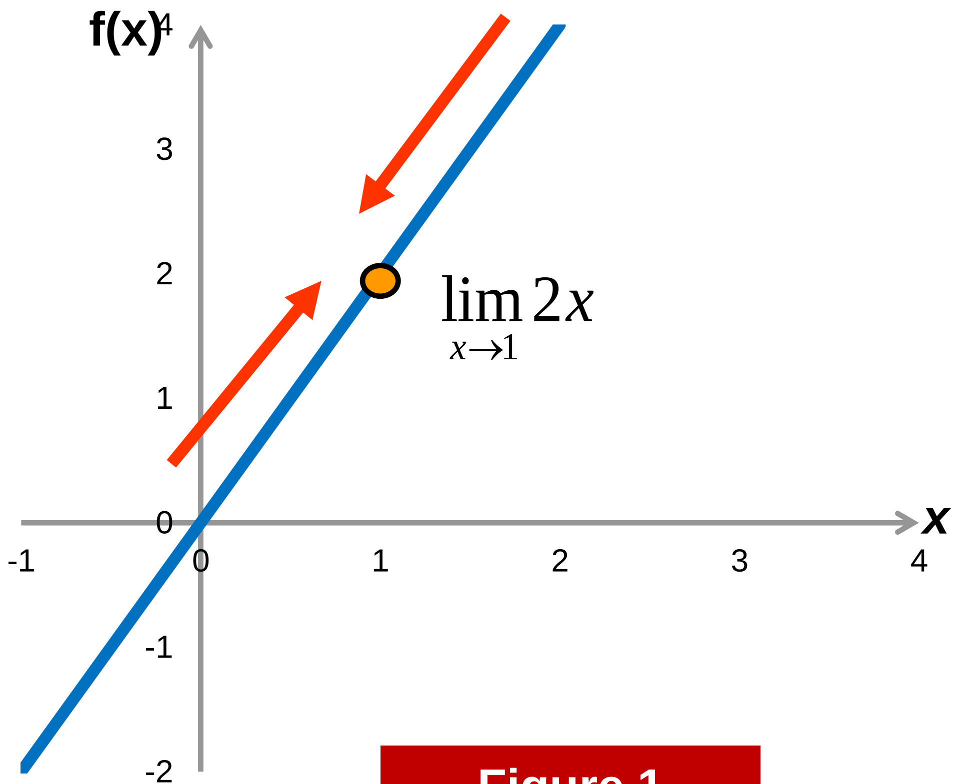


Figure 1

Example

Evaluate $\lim_{x\to 1}(x+2)^2$ graphically.

This function is a quadratic function where $y=(x+2)^2$. As we can see in **the Figure 2**, f(x) approached 9 when *x* comes closer to 1 from both sides.

$$\lim_{x\to 1^-}(x+2)^2=\lim_{x\to 1^-}(x+2)^2=9$$

$$\therefore \lim_{x\to 1}(x+2)^2=9$$

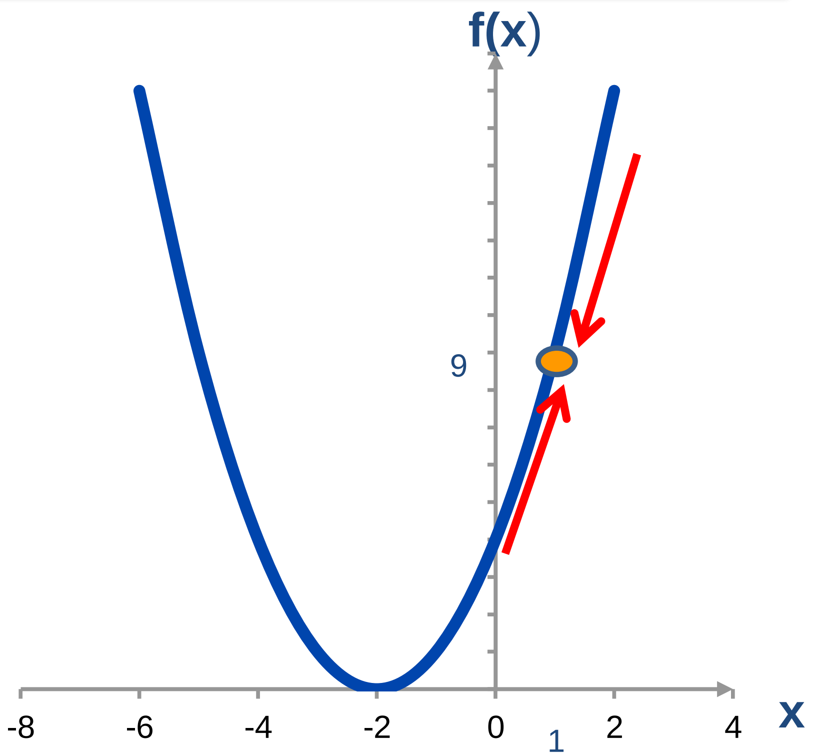


Figure 2

Example

Given $f(x)=\begin{cases} x+1, & x<1 \\ \frac{3}{2}, & x=1 \\ 2-x, & x>1 \end{cases}$. Find $\lim\limits_{x\to 1} f(x)$ graphically.

We choose $y=x+1$ for the left hand side function, meanwhile right hand side function is $y=2-x$. The movement of the graph tells us that

$$\lim_{x\to 1^-} f(x)=2 \qquad \lim_{x\to 1^+} f(x)=1$$

Since the value of one sided limit is different

$$\therefore \lim_{x\to 1} f(x) \text{ does not exist}$$

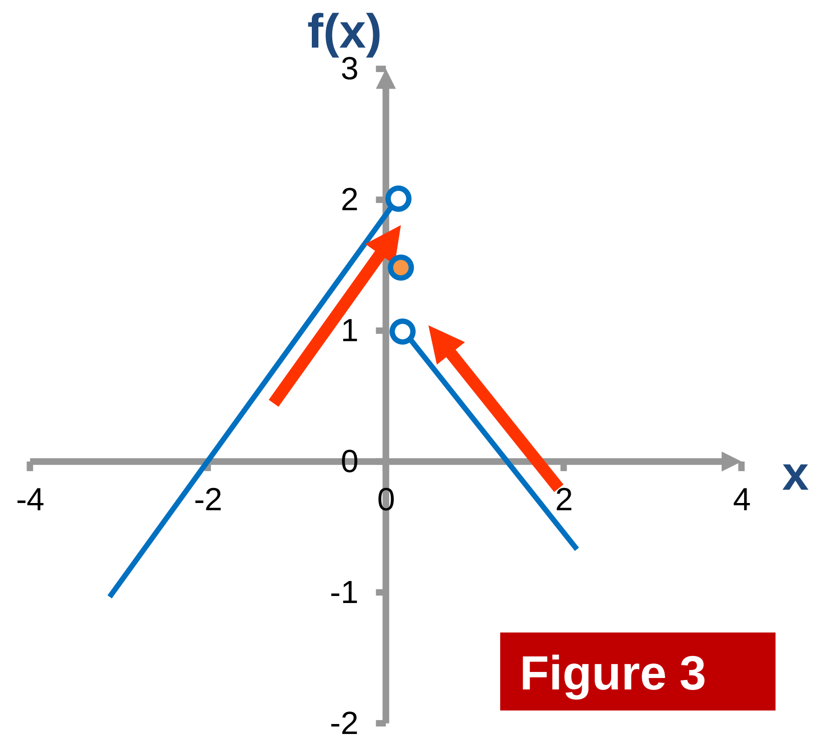


Figure 3

Example

The diagram below shows the graph of the function, *f*. Find

(a) $\lim_{x \to 2^-} f(x)$ (b) $\lim_{x \to 2^+} f(x)$ (c) $\lim_{x \to 2} f.(x)$

(d) $\lim_{x \to 4^-} f(x)$ (e) $\lim_{x \to 4^+} f(x)$ (f) $\lim_{x \to 4} f(x)$

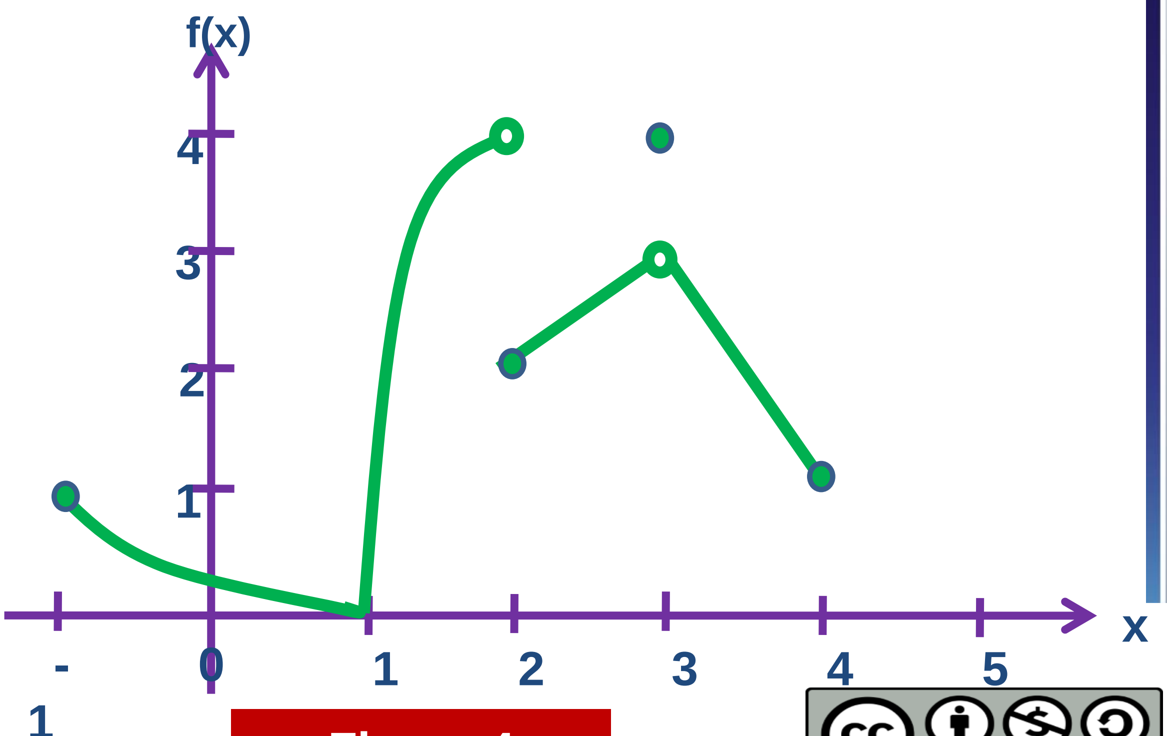


Figure 4

(a) $\lim_{x \to 2^-} f(x) = 4$

(b) $\lim_{x \to 2^+} f(x) = 2$

(c) $\lim_{x \to 2} f(x)$ does not exist

(d) $\lim_{x \to 4^-} f(x) = 1$

(e) $\lim_{x \to 4^+} f(x)$ does not exist

(f) $\lim_{x \to 4} f(x)$ does not exist

Analytical Method

Limits law will be used extensively in solving limit problem. If the limit cannot be evaluated by limit laws (1), then the algebraic technique (2) will be used

1) Limits Law Technique

- Substitutions

2) Algebraic Technique

- Factorization
- Multiplication of conjugate

indeterminate form

Limits Law

Constant Rule

$$\lim_{x\to c} k = k$$

Identity Rule

$$\lim_{x\to c} x = c$$

Sum and Difference Rule

$$\lim_{x\to c}(f(x)\pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$$

Product Rule

$$\lim_{x\to c}(f(x).g(x)) = \lim_{x\to c} f(x).\lim_{x\to c} g(x)$$

Constant Multiple Rule

$$\lim_{x\to c}(k.f(x)) = k.\lim_{x\to c} f(x)$$

Quotient Rule

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \rightarrow \lim_{x\to c} g(x) \neq 0$$

Power Rule

$$\lim_{x\to c}(f(x))^n = \left(\lim_{x\to c} f(x)\right)^n$$

1. Constant Rule

- The limit of a constant is the constant itself

$$\lim_{x \to 2} 6 = 6$$

$$\lim_{x \to 3} -11 = -11$$

2. Identity Rule

- The limit of function f(x), where f(x)=x, is c since x approaches c.

$$\lim_{x \to 2} x = 2$$

$$\lim_{x \to -7} x = -7$$

3. Sum and Difference Rule

- The limit of the sum of two functions is the sum of their limits

$$\lim_{x \to 2} (x+4) = 2+4 = 6$$

$$\lim_{x \to 3} (x-4) = 3-4 = -1$$

4. Product Rule

- The limit of a product of two functions is the product of their limits.

$$\lim_{x\to 2}(x+1)(x+2)$$
$$=\lim_{x\to 2}(x+1)\cdot\lim_{x\to 2}(x+2)$$
$$=3\cdot 4=12$$

5. Constant Multiple Rule

- The limit of a constant, multiply by a function is the constant multiply by the limits of the function

$$\lim_{x\to 2}2(4x-1)$$
$$=2\lim_{x\to 2}(4x-1)$$
$$=2(7)=14$$

6. Quotient Rule

- The limit of quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero

$$\lim_{x\to 3}\left(\frac{x-1}{x+3}\right)=\frac{\lim_{x\to 3}(x-1)}{\lim_{x\to 3}(x+3)}$$
$$=\frac{3-1}{3+3}=\frac{1}{3}$$

7. Power Rule

- The limit of the nth power is the nth power of the limit where *n* is a positive integer and

$$f(c) > 0$$

$$\lim_{x \to c} (f(x))^n = \left(\lim_{x \to c} f(x) \right)^n$$

$$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$$

$$\lim_{x \to 1} (3x)^2 = \left(\lim_{x \to 1} (3x) \right)^2 = (3)^2 = 9$$

$$\lim_{x \to -2} \sqrt{5x^2 - 4} = \sqrt{\lim_{x \to -2} (5x^2 - 4)} = \sqrt{20 - 4} = 4$$

Example

Evaluate the following limit analytically.

(a) $\lim_{x\to 1}(x+5)$ (b) $\lim_{x\to -2}(3x+5)$ (c) $\lim_{x\to 1} 2x(x+4)$

(d) $\lim_{x\to 2}\left(\frac{3x+4}{x+3}\right)$ (e) $\lim_{x\to 4}\left(\sqrt{x^2+11}\right)^{1/3}$ (f) $\lim_{x\to 2}\frac{x^2+x-6}{x-2}$

(a) $\lim_{x\to 1}(x+5)=1+5=6$

(b) $\lim_{x\to -2}(3x+5)=3(-2)+5=-1$

(c) $\lim_{x\to 1} 2x(x+4)=2(1+4)=10$

(d) $\lim_{x\to 2}\left(\frac{3x+4}{x+3}\right)=\frac{3(2)+4}{2+3}=\frac{10}{5}=2$

(e) $\lim_{x\to 4}\left(\sqrt{x^2+11}\right)^{1/3}=\left(\sqrt{(4)^2+11}\right)^{1/3}=\sqrt{3}$

(f) $\lim_{x\to 2}\frac{x^2+x-6}{x-2}=\frac{0}{0}$

indeterminate form

For case (f), direct substitution doesn't always work!

- There are cases that we cannot solve using the limit laws technique.

$$\text{e.g. } \lim_{x \to 2}\left[\frac{x^2 + x - 6}{x - 2}\right] = \frac{0}{0}$$

- If $\lim_{x \to c} f(x) = \frac{0}{0}$, it cannot be evaluated by direct substitution.
- Use Algebraic technique instead such as ;

Factoring

Multiplying conjugate

Example

Evaluate the following limit analytically.

(a) $\lim\limits_{x\to 2}\dfrac{x^2+x-6}{x-2}$ (b) $\lim\limits_{t\to 1}\dfrac{t^2-1}{t-1}$ (c) $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}$

(d) $\lim\limits_{t\to 0}\dfrac{\sqrt{3+t}-\sqrt{3}}{t}$ (e) $\lim\limits_{x\to 1}\dfrac{3x^2-1}{x+2}$ (f) $\lim\limits_{p\to 2}\dfrac{1}{\sqrt{4p^2-7}}$

(a) $$\lim_{x\to 2}\frac{x^2+x-6}{x-2}=\lim_{x\to 2}\frac{(x+3)(x-2)}{x-2}=5$$

(b) $$\lim_{t\to 1}\frac{t^2-1}{t-1}=\lim_{t\to 1}\frac{(t-1)(t+1)}{t-1}=2$$

(c) $$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}=\frac{1}{4}$$

(d) $$\lim_{x\to 0}\frac{\sqrt{3+t}-\sqrt{3}}{t}=\frac{1}{2\sqrt{3}}$$

(e) $$\lim_{x\to 1}\frac{3x^2-1}{x+2}=\frac{2}{3}$$

(f) $$\lim_{p\to 2}\frac{1}{\sqrt{4p^2-7}}=\frac{1}{3}$$

Conclusions #1

- The **notation** $\lim_{x\to c} f(x) = L$ is read " the limit $f(x)$ as x approaches c is L" and means that the functional values $f(x)$ can be made arbitrarily, close to L by choosing x sufficiently close to c (but not equal to c).
- L is a finite real number. If L can be found, then the limit of exists. If L cannot be found or infinite, then the limit of $f(x)$ **does not exist.**
- Numerical method: use table to calculate the limit (i.e. consider limit from both sides)
- Graphical method : use graph to determine the limit (i.e. consider limit from both sides)

Conclusions #2

- Analytical method: use properties to determine the limit
- $\lim_{x\to c} f(x) = \frac{0}{0}$ or $\lim_{x\to c} f(x) = \frac{\infty}{\infty}$ cannot be evaluated by direct substitution or using properties of limit . This kind of form must be solve using either of the following method. **Factorisation, multiplying of conjugate or fraction reduction.**

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