FACULTY OF INDUSTRIAL SCIENCES & TECHNOLOGY
FINAL EXAMINATION

COURSE	**:**	**NUMERICAL METHODS**
COURSE CODE	**:**	**BUM2313**
LECTURER	**:**	**NAWWARAH BINTI SUHAIMY** **NADIRAH BINTI MOHD NASIR** **NORHAYATI BINTI ROSLI**
DATE	**:**	**30 DECEMBER 2015**
DURATION	**:**	**3 HOURS**
SESSION/SEMESTER	**:**	**SESSION 2015/2016 SEMESTER I**
PROGRAMME CODE	**:**	**BAA/BAE/BEC/BEE/BEP/BMA/BMB/BMF/ BMI/BMM**

INSTRUCTIONS TO CANDIDATES:

1. This question paper consists of **SIX (6)** questions. Answer **ALL** questions.
2. Use **FOUR (4) decimal places** in all calculations **EXCEPT for Question 6(a)**, use **SEVEN (7) decimal places**.
3. All answers to a new question should start on a new page.
4. All the calculations and assumptions must be clearly stated.
5. Candidates are not allowed to bring any material other than those allowed by the invigilator into the examination room.

EXAMINATION REQUIREMENTS

1. Scientific Calculator
2. **APPENDIX**

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO

This examination paper consists of **ELEVEN (11)** printed pages including front page.

QUESTION 1

Given the system of linear equations

$$34x_1 + 12x_2 + 15x_3 = 82$$
$$12x_1 + 16x_2 + 17x_3 = 69$$
$$15x_1 + 17x_2 + 22x_3 = 92.$$

(i) Transform the above system of linear equations in matrix form, $AX = b$.

(1 Mark)

(ii) Solve the system of linear equations by using Cholesky factorization.

(17 Marks)

[18 Marks, CO2/PO2]

QUESTION 2

Solve the system of two nonlinear equations

$$y - x^2 + 2 = 0$$
$$x^2 + (y-3)^2 - 9 = 0$$

using Newton-Raphson method with ONE iteration and an initial guess of $x_0 = 1.6$ and $y_0 = 7$.

[12 Marks, CO2/PO2]

QUESTION 3

A simple pendulum consists of a mass that swings in a vertical plane at the end of a massless rod of length *L*, as shown in **Figure 1**.

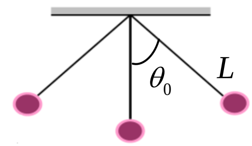

Figure 1

Suppose that a simple pendulum is displaced through an angle, θ_0 and released from rest. It can be shown that in the absence of friction, the time for the pendulum to make one complete back and forth swing, called the period is given by

$$T = \sqrt{\frac{8L}{g}} \int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}} \, d\theta \qquad (1)$$

where $\theta = \theta(t)$ is the angle the pendulum makes with the vertical at time, t. The integral (1) is difficult to evaluate. By a substitution outlined below

$$\cos\theta = 1 - 2\sin^2\left(\theta/2\right), \ \cos\theta_0 = 1 - 2\sin^2\left(\theta_0/2\right) \text{ and } k = \sin\left(\theta_0/2\right),$$

and the change of variable

$$\sin\phi = \frac{\sin\left(\theta/2\right)}{\sin\left(\theta_0/2\right)} = \frac{\sin\left(\theta/2\right)}{k},$$

the period, T that expressed by equation (1) can be written as

$$T = 4\sqrt{\frac{L}{g}} \left(\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2\phi}} \, d\phi \right). \qquad (2)$$

The integral (2) is called a complete elliptic integral of the first kind. Estimate the period of the simple pendulum given in (2) by using Simpson's rule if $L = 1.5$ ft, $g = 32 \text{ ft/s}^2$, $k = 0.3827$ and the number of strips, $n = 9$. (*Hint: Use radian mode in your calculator*)

[12 Marks, CO2/PO2]

QUESTION 4

The mixture problem of brine solution which describe the amount of salt, $Q(t)$, in the tank after t minutes is given by the following first linear order differential equations

$$\frac{dQ}{dt}+\left(\frac{2}{200+t}\right)Q=\frac{9}{5}\left(1+\cos(t)\right).$$

The tank initially contains 5 litres of salt. Find the amount of salt in the tank for $0 \le t \le 300$ using

(i) second order Runge-Kutta of Heun method with a step size of 150 minutes.

(9 Marks)

(ii) fourth order Runge-Kutta method with a step size of 300 minutes. Then, calculate its true percent relative error if the analytical solution of this rate of change can be determined using linear method as

$$Q(t)=\frac{9}{5}\left(\frac{1}{3}(200+t)+\sin(t)+\frac{2\cos(t)}{200+t}-\frac{2\sin(t)}{(200+t)^2}\right)-\frac{4600720}{(200+t)^2}.$$

(*Hint: Use radian mode in your calculator*)

(8 Marks)

[17 Marks, CO2/PO2]

QUESTION 5

The reactivity behaviour of porous catalyst particles subject to both internal mass concentration gradients is expressed as

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \Phi^2 y e^{\phi} \quad \text{with } y(0.5) = 0.3 \text{ and } y(1.0) = 1.0$$

where

$$\phi = \frac{\gamma\beta(1-y)}{1+\beta(1-y)}$$

y: dimensionless concentration
x: dimensionless radial coordinate (spherical geometry)
Φ: Thiele modulus (first-order reaction rate)
γ: Arrhenius number
β: Prater number

Find the solutions of the above equation by using Shooting method where $\gamma = 30$, $\beta = 0.4$ and $\Phi = 0.3$. Use a step size, $h = 0.25$ with the first guess $z(0.5) = 2.0$ and second guess $z(0.5) = -2.0$.

[18 Marks, CO2/PO2]

QUESTION 6

Consider a simply supported beam with a modulus of elasticity, E, moment of inertia, I, a uniform load, w, length, L, and end tension, T as illustrated in **Figure 2**.

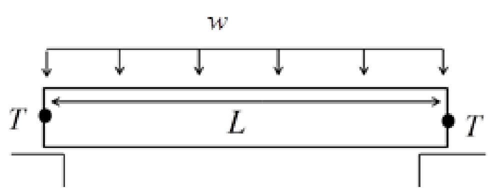

Figure 2

If $y(x)$ denotes the deflection at each point x in the beam, and for the situation where the deflection is small, then the equation of deflection of the beam can be approximated by the linear second order ordinary differential equations as

$$\frac{1}{T}y'' - \frac{1}{EI}y = \frac{Twx(L-x)}{2EI}$$

with boundary conditions, $y(0) = y(L) = 0$. Suppose the beam is a W12×12 structural steel I-beam with $L = 100$ in, $E = 29\times10^6$ lb/in^2, $I = 121$ in^4, $T = 10,000$ lb and the beam is carrying a uniform load of $w = 10,000$ lb.

(i) Perform discretization process for the above linear second order differential equations by dividing the length of the beam into five equal subintervals.

(3 Marks)

(ii) Using your discretization output, write y_i at each interior nodes, x_i. Hence, transform your system of linear equations into tridiagonal matrices.

(7 Marks)

(iii) Find the approximate solution of the interior deflection as given by tridiagonal matrices in part (ii) using Thomas algorithm method.

(13 Marks)

[23 Marks, CO2/PO2]

END OF QUESTION PAPER

APPENDIX

Errors

True Error

E_t = true value - approximation value

True percent relative error

$$\varepsilon_t = \left|\frac{\text{true value} - \text{approximation value}}{\text{true value}}\right| \times 100$$

Approximate percent relative error

$$\varepsilon_a = \left|\frac{\text{present approximation} - \text{previous approximation}}{\text{present approximation}}\right| \times 100$$

Stopping criterion

Terminate computation when $|\varepsilon_a| < \varepsilon_s$

Roots of Equations

Bisection method

$$x_r = \frac{(x_l + x_u)}{2}$$

False-position method

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

Secant method

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

Newton-Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Linear Algebraic Equations and Matrices

System of linear algebraic equations

$[A]\{X\} = \{B\}$. Decomposition $[A] = [L][U]$ with $[L]$ and $[U]$ can be obtained as follows:

Using Doolittle decomposition

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}; \; [U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Using Cholesky method

$$[A] = [U]^T[U] \; ; \quad [U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix},$$

$$u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2}$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} u_{ki}u_{kj}}{u_{ii}} \quad \text{for } j = i+1, \ldots, n$$

Using Crout's method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Jacobi method

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \left(\sum_{\substack{j=1 \\ j\neq i}}^{j=n} a_{ij} x_j^{(k)} \right) \right], i = 1, 2, ..., n.$$

Gauss seidel method

$$x_i^{(k+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{(k+1)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(k)}$$

where

$k = 1, 2, \ldots$

$i = 1, 2, \ldots, n$

Power method

$$v^{(k+1)} = \frac{1}{m_{k+1}} A v^{(k)}$$

$k = 0,1,2,....$

Nonlinear system: Newton-Raphson method

$$x_{1,i+1} = x_{1,i} - \frac{f_{1,i}\dfrac{\partial f_{2,i}}{\partial x_2} - f_{2,i}\dfrac{\partial f_{1,i}}{\partial x_2}}{|J|}$$

$$x_{2,i+1} = x_{2,i} - \frac{f_{2,i}\dfrac{\partial f_{1,i}}{\partial x_1} - f_{1,i}\dfrac{\partial f_{2,i}}{\partial x_1}}{|J|}$$

$$[J] = \begin{bmatrix} \dfrac{\partial f_{1,i}}{\partial x_1} & \dfrac{\partial f_{1,i}}{\partial x_2} \\ \dfrac{\partial f_{2,i}}{\partial x_1} & \dfrac{\partial f_{2,i}}{\partial x_2} \end{bmatrix}$$

Curve Fitting

Newton interpolation polynomial

$$f_n(x) = f(x_0) + f[x_0, x_1](x - x_0)$$
$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ \cdots + f[x_0, \ldots, x_{n-1}, x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

where

$$f[x_0, \ldots, x_{n-1}, x_n] = \frac{f[x_n, x_{n-1}, \ldots, x_1] - f[x_{n-1}, x_{n-2}, \ldots, x_0]}{x_n - x_0}$$

Lagrange interpolation polynomial

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i) \text{ where } L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$n -$ order of interpolation

Inverse Newton interpolation polynomial

$$P_n(f) = x_0 + b_1(f - f_0) + b_2(f - f_0)(f - f_1) + b_3(f - f_0)(f - f_1)(f - f_2) + \cdots + b_n(f - f_0)(f - f_1)\cdots(f - f_{n-1})$$

$n -$ order of interpolation

Inverse Lagrange interpolation polynomial

$$P_n(f) = \sum_{i=0}^{n} L_i(f)x_i \quad \text{where} \quad L_i(f) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(f - f_j)}{(f_i - f_j)}$$

$n -$ order of interpolation

Linear Splines

$$s_i(x) = f(x_i) + \frac{f_{i+1} - f_i}{x_{i+1} - x_i}(x - x_i) \qquad x_i \le x \le x_{i+1}$$

Quadratic Splines

$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 \qquad x_i \le x \le x_{i+1}$$

For $i = 1, 2, \ldots, n-1$, find

$h_i = x_{i+1} - x_i; \qquad f_i + b_i h_i + c_i h_i^2 = f_{i+1}$

$b_i + 2c_i h_i = b_{i+1};$

Also given,

$c_1 = 0$

$a_i = f_i$

Numerical Integration

Trapezoidal rule

$$I \cong \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right]$$

where

$$h = \frac{x_n - x_0}{n}$$

Simpson's 1/3rd rule

$$I \cong \frac{h}{3}\left[f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n) \right]$$

where

$h = \dfrac{x_n - x_0}{n}$ and n must even segment

Simpson's 3/8 rule

$$I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right], \text{ where } h = \frac{x_3 - x_0}{3}$$

Ordinary Differential Equations (IVP)

Euler's method

$$y_{i+1} = y_i + hf(x_i, y_i)$$

$$x_{i+1} = x_i + h$$

2nd order Runge-Kutta: Heun method

$$y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)h$$
$$x_{i+1} = x_i + h$$
where
$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + h, y_i + k_1 h)$$

2nd order Runge-Kutta: Midpoint method

$$y_{i+1} = y_i + k_2 h$$
$$x_{i+1} = x_i + h$$
where
$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

2nd order Runge-Kutta: Ralston's method

$$y_{i+1} = y_i + \frac{1}{3}(k_1 + 2k_2)h$$
$$x_{i+1} = x_i + h$$

where
$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h)$$

Fourth order Runge-Kutta method

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$x_{i+1} = x_i + h$$
where
$$k_1 = f\left(x_i, y_i\right)$$
$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$
$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$
$$k_4 = f\left(x_i + h, y_i + k_3 h\right)$$

Ordinary Differential Equations (BVP)
Shooting method Extrapolate estimate for initial slope $z(0) = G1 + \frac{G2 - G1}{R2 - R1}(D - R1)$ where G1 = First guess at initial slope G2 = Second guess at initial slope R1 = Final result at endpoint (using G1) R2 = Second result at endpoint (using G2) D = the desired value at the endpoint
Finite Difference method $\frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}$ $\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2\Delta x}$ $\left(1 - \frac{\Delta x}{2} p_i\right) y_{i-1} - \left(2 - \Delta x^2 q_i\right) y_i + \left(1 + \frac{\Delta x}{2} p_i\right) y_{i+1} = \Delta x^2 r_i$ **Thomas Algorithm** $\alpha_i = d_i - c_i \beta_{i-1} \quad , \quad \alpha_1 = d_1$ $\beta_i = \frac{e_i}{\alpha_i}$ $w_i = \frac{b_i - c_i w_{i-1}}{\alpha_i} \quad , \quad w_1 = \frac{b_1}{\alpha_1}$ $y_i = w_i - \beta_i y_{i+1} \quad , \quad y_4 = w_4$