Numerical Methods
Ordinary Differential Equations: Boundary Value Problems (BVP)

by

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Description

AIMS

This chapter is aimed to solve boundary value problems of second order ODEs by using two different types of methods involving shooting method and finite difference method.

EXPECTED OUTCOMES

Students should be able to solve boundary value problems using shooting method and finite difference method.

REFERENCES

1. Norhayati Rosli, Nadirah Mohd Nasir, Mohd Zuki Salleh, Rozieana Khairuddin, Nurfatihah Mohamad Hanafi, Noraziah Adzhar. *Numerical Methods,* Second Edition, UMP, 2017 (Internal use)
2. Chapra, C. S. & Canale, R. P. *Numerical Methods for Engineers*, Sixth Edition, McGraw–Hill, 2010.

Content

1 Introduction to Ordinary Differential Equations (Boundary Value Problems)

2 Numerical Methods of ODEs (BVP)

2.1 Shooting Method

2.2 Finite Difference Method

INTRODUCTION

- In initial value problem, the condition is specified at the same value of the independent variable.
- However, for boundary value problem (BVP), the conditions are specified at different values of the independent variables.
- In a nutshell, a BVP is a problem, typically ODE which has values assigned on the physical boundary of the domain.

General Form of ODEs (BVP)

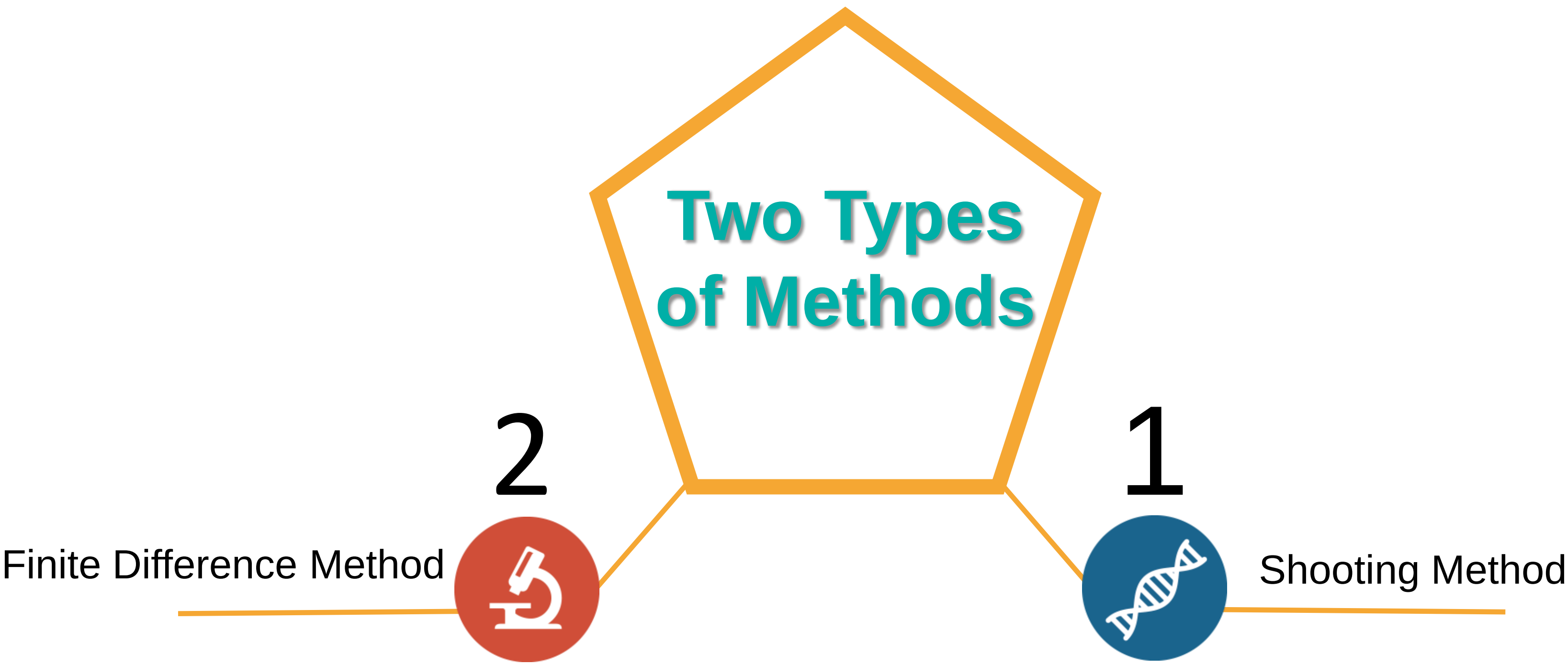
$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad x \in [a, b]$$

$$y(a) = \alpha, \qquad y(b) = \beta$$

INTRODUCTION (Cont.)

- Most of ODEs (BVP) cannot be solved analytically.
- Its due to the complexity form of the equations.
- Numerical methods offer a viable option to solve ODEs (BVP).

Numerical Methods for Solving ODEs (IVP)



SHOOTING METHOD

Introduction

- Shooting method is a technique of converting the boundary value problem to an equivalent initial value problem.
- Then, an initial value problem is solved via a trial and error approach. This technique is called a "shooting" method, by analogy to the procedure of shooting the object at a stationary target.

SHOOTING METHOD (Cont.)

Shooting Method Procedures

Step 1

Reduce the second order ODE (BVP) of equation to a system of first order ODE (IVP). The second order ODE is transformed into a system of two first order ODEs as

$$f_1(x, y, z) = \frac{dy}{dx} = z, \quad y(x_0) = y_0 \quad \text{or} \quad y(a) = \alpha$$

$$f_2(x, y, z) = \frac{dz}{dx} = \frac{d^2 y}{dx^2}, \quad z(x_0) = z_0$$

Step 2

Determine the initial value. The boundary value at the first point of the domain is known and is used as one initial value of the system. The additional initial value that required for solving the system is guessed.

SHOOTING METHOD (Cont.)

Shooting Method Procedures (Cont.)

Step 3

The equivalent system of initial value problem is then solved via Euler's method, RK2 method or RK4 method. However, in this course only Euler's method shall be considered.

Step 4

The solution obtained at the end point of the domain is compared with the boundary condition. If the numerical solution is differ from the boundary condition, the guess initial value is changed, and the system is solved again.

SHOOTING METHOD (Cont.)

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Shooting Method Procedures (Cont.)

Step 5

If the result obtained from the second initial guess also differ from the boundary conditions, extrapolate the initial value using linear extrapolate formula

$$z(0) = G1 - \frac{G2 - G1}{R2 - R1}(D - R1)$$

where

$G1$: First guess at initial slope

$G2$: Second guess at initial slope

$R1$: Final result at endpoint (using $G1$)

$R2$: Second result at endpoint (using $G2$)

D : The desired value at the endpoint

The equivalent system of initial value problem is then solved via Euler's method.

SHOOTING METHOD (Cont.)

Example 1

Use the Shooting method to approximate the solution of the boundary value problem

$$y''(x) - 2y(x) = 0, \quad y(0) = 1.2, \quad y(1.0) = 0.9, \quad h = 0.25$$

Let the first guess, $z_0 = -1.5$ and the second guess, $z_0 = -1.0$

Solution

Step 1

Reduce the second order ODE (BVP) to a system of first order ODE (IVP).

$$f_1(x, y, z) = \frac{dy}{dx} = z$$

$$f_2(x, y, z) = \frac{dz}{dx} = 2y$$

SHOOTING METHOD (Cont.)

Solution (Cont.)

Step 2

Given $x_0 = 0, y_0 = 1.2, z_0$, approximate the system of first order ODE via Euler method.

i	x	$y(x)$	$z(x)$
0	0.0000	1.2000	-1.5000
1	0.2500	0.8250	-0.9000
2	0.5000	0.6000	-0.4875
3	0.7500	0.4781	-0.1875
4	1.0000	0.4312	0.0516

SHOOTING METHOD (Cont.)

Solution (Cont.)

Step 3

$y(1.0) \approx 0.4312 \neq 0.9$. Use a second guess, $z_0 = -1.0$ and approximate the system of first order ODE via Euler method.

i	x	$y(x)$	$z(x)$
0	0.0000	1.2000	-1.0000
1	0.2500	0.9500	-0.4000
2	0.5000	0.8500	0.0750
3	0.7500	0.8688	0.5000
4	1.0000	0.9938	0.9344

SHOOTING METHOD (Cont.)

Solution (Cont.)

Step 4

$y(1.0) \approx 0.9938 \neq 0.9$. Extrapolate the initial value using linear extrapolate formula

$$z(0) = -1.5 + \frac{-1.0-(-1.5)}{0.9938-0.4312}(0.9-0.4312)$$
$$= -1.0834$$

SHOOTING METHOD (Cont.)

Solution (Cont.)

Step 5

The equivalent system of initial value problem is then solved via Euler's method

i	x	$y(x)$	$z(x)$
0	0.0000	1.2000	-1.0834
1	0.2500	0.9292	-0.4834
2	0.5000	0.8083	-0.0188
3	0.7500	0.8036	0.3854
4	1.0000	0.9000	0.7872

Since, $y_4 = y(1.0) = 0.9$ (same with boundary condition), the solutions that were obtained in Step 5 is acceptable.

Therefore, $y(0) = 1.2$; $y(0.25) \approx 0.9292$; $y(0.5) \approx 0.8083$; $y(0.75) \approx 0.8036$ and $y(1.0) = 0.9$.

FINITE DIFFERENCE METHOD

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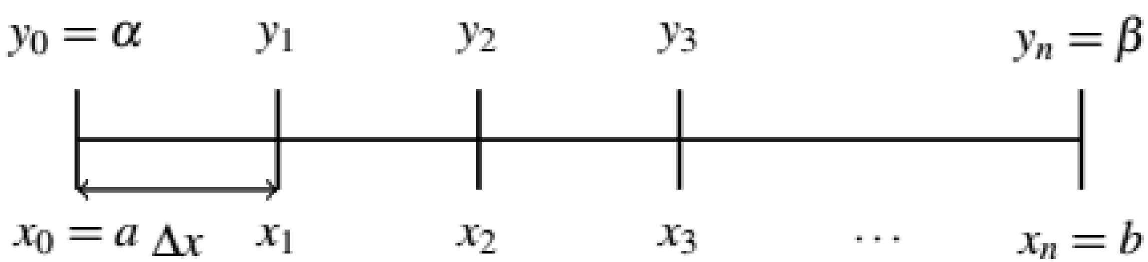
General Form of Second Order ODEs (BVP)

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x)$$

$$y(a) = \alpha, \qquad y(b) = \beta$$

- Interval $[a, b]$ is divided into n subinterval with a step size Δx.



FINITE DIFFERENCE METHOD (Cont.)

- Boundary values y_0 and y_n are known.
- The approximate values of y_i for $i = 1,2, \ldots, n-1$ is computed.
- In finite difference method, a general form of second order ODE at the interior mesh points $x = x_i$ for $i = 1,2, \ldots, n-1$.

General Form Second Order ODEs

$$y''(x_i) + p(x_i)y'(x_i) + q(x_i)y(x_i) = r(x_i)$$

or

$$y''_i + p_i y'_i + q_i y_i = r_i.$$

FINITE DIFFERENCE METHOD (Cont.)

Finite Difference Method: Formula

- By using central difference formulas for first and second derivatives:

$$\frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}$$

$$\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2\Delta x}$$

- Substituting the first and second derivatives in general form of second order ODEs (BVP)

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} + p_i \frac{y_{i+1} - y_{i-1}}{2\Delta x} + q_i y_i = r_i.$$

FINITE DIFFERENCE METHOD (Cont.)

Finite Difference Method: Formula (Cont.)

- Multiplying both sides of equation (1) and with some algebraic simplification, yields

Finite Difference Formula

$$y_{i+1} - 2y_i + y_{i-1} + \frac{\Delta x}{2} p_i \left(y_{i+1} - y_{i-1} \right) + \Delta x^2 q_i y_i = \Delta x^2 r_i$$

or rearrange

$$\left(1 - \frac{\Delta x}{2} p_i \right) y_{i-1} - \left(2 - \Delta x^2 q_i \right) y_i + \left(1 + \frac{\Delta x}{2} p_i \right) y_{i+1} = \Delta x^2 r_i$$

for $i = 1, 2, 3, \ldots, n-1.$

- Applying finite difference formula for each interior nodes leads to a **set of simultaneous algebraic equations.**

FINITE DIFFERENCE METHOD (Cont.)

Finite Difference Method: Tridiagonal System

- Transforming the **simultaneous algebraic equations** into **Tridiagonal system**

$$\begin{bmatrix} d_1 & e_1 & 0 & \dots & 0 \\ c_2 & d_2 & e_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & c_{n-1} & \dots & d_{n-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

where d_i, e_i and c_i for $i = 1,2,\dots,n-1$ are the coefficients of $y_1, \dots, y_{n-1}$.

FINITE DIFFERENCE METHOD (Cont.)

Finite Difference Method: Thomas Algorithm

- Applying **Direct Methods of solving linear algebraic equations or Thomas Algorithm**

Thomas Algorithm Formula

$$\alpha_i = d_i - c_i \beta_{i-1}$$

$$\alpha_1 = d_1$$

$$\beta_i = \frac{e_i}{\alpha_i}$$

$$w_i = \frac{b_i - c_i w_{i-1}}{\alpha_i}$$

$$w_1 = \frac{b_1}{\alpha_1}$$

$$y_i = w_i - \beta_i y_{i+1}$$

$$y_{n-1} = w_{n-1}$$

FINITE DIFFERENCE METHOD (Cont.)

Finite Difference Method Procedures

Step 1 Write the second order ODEs (BVP) in general form

Step 2 Identifying p, q and r

Step 3 Compute the step size, $\Delta x = \frac{x_n - x_0}{n}$ by performing the discretization process. Substitute p, q and r and Δx into equation.

FINITE DIFFERENCE METHOD (Cont.)

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Finite Difference Method Procedures

Step 4 Use the discretization output in Step 3 to write y_i at each x_i

Step 5 Transform the system of linear equations into tridiagonal system

Step 6 Solve the tridiagonal system by using **Thomas algorithm**

FINITE DIFFERENCE METHOD (Cont.)

Example 2

Use the Finite Difference method to approximate the solution of the boundary value problem

$$y''(x) - 2y(x) = 0, \quad y(0) = 1.2, \quad y(1.0) = 0.9, \quad h = 0.25$$

Solution

Step 1

Write the second order ODE (BVP) in general form

$$y'' - 2y = 0$$

Step 2

$$p_i = 0,\ q_i = -2,\ r_i = 0$$

FINITE DIFFERENCE METHOD (Cont.)

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Solution (Cont.)

Step 3

Substitute p, q, r and Δx into finite difference formula

$$\left(1-\frac{0.25}{2}(0)\right)y_{i-1}-\left(2-0.25^2(-2)\right)y_i+\left(1+\frac{0.25}{2}(0)\right)y_{i+1}=0.25^2(0)$$

The finite difference formula is

$$y_{i-1}-2.125y_i+y_{i+1}=0,\ \ i=1,2,3$$

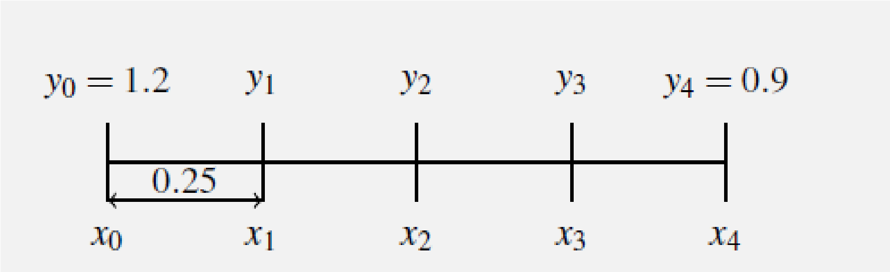
FINITE DIFFERENCE METHOD (Cont.)

Solution (Cont.)

Step 4

Discretization process

$$n = \frac{1-0}{0.25} = 4$$



Write y_i at each interior nodes, x_i

FINITE DIFFERENCE METHOD (Cont.)

Solution (Cont.)

Step 5

Write y_i at each interior nodes, x_i

For $i = 1, x_1 = 0.25$

$$y_0 - 2.125y_1 + y_2 = 0$$
$$-2.125y_1 + y_2 = -1.2$$

For $i = 2, x_2 = 0.5$

$$y_1 - 2.125y_2 + y_3 = 0$$

For $i = 3, x_3 = 0.75$

$$y_2 - 2.125y_3 + y_4 = 0$$
$$y_2 - 2.125y_4 = -0.9$$

FINITE DIFFERENCE METHOD (Cont.)

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Solution (Cont.)

Step 6

Tridiagonal system

$$\begin{bmatrix} -2.125 & 1 & 0 \\ 1 & -2.125 & 1 \\ 0 & 1 & -2.125 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

FINITE DIFFERENCE METHOD (Cont.)

Solution (Cont.)

Step 7

Approximate the solution by using **Thomas Algorithm**

i	1	2	3
b_i	-1.2	0	-0.9
c_i	–	1	1
d_i	-2.125	-2.125	-2.125
e_i	1	1	–
α_i	-2.125	-1.6544	-1.5206
β_i	-0.4706	-0.6044	–
w_i	0.5647	0.3413	0.8163
y_i	0.9575	0.8347	0.8163

Therefore the approximated solutions to the boundary value problems are
$y_1 = 0.9575, y_2 = 0.8347, y_3 = 0.8163$

Conclusion

- The shooting technique for BVP can present problem of instability.
- The most common alternative to the shooting method is finite difference approach.
- This method has better stability characteristics than shooting method.
- However, it requires more computation to obtain a specified accuracy.

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