9. Ordinary Differential Equations: Initial Value Problem

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9.1 Exercises

Euler Method

Exercise 9.1 Estimate the following initial value problems by using Euler's method. Use a step size of $h = 0.5$.

i. $y' = y(sin(t))^5, \qquad 0 \le t \le 2, \quad y(0) = 1$

ii. $y' = yx + \exp(-xy), \qquad 0 \le x \le 2, \quad y(0) = 1$

iii. $y' = y + t^2 \sin(y), \qquad 0 \le t \le 3, \quad y(0) = 1.5$

iv. $y' = 0.4\left(1 - \frac{y}{3.5}\right)y, \qquad 0 \le x \le 5, \quad y(0) = 0.031$

v. $y' = -\exp\left(\frac{10y}{573}\right), \qquad 0 \le x \le 2, \quad y(0) = 1$

Exercise 9.2 The open loop for the speed, w, to a voltage input of $20V$ is

$$(0.2)\frac{dw}{dt} = 10 - 0.6w.$$

where t is a time measured in hour. If the initial speed is zero, calculate the speed at $t = 1.0$ with a step size of 0.25 by using Euler method.

Exercise 9.3 Let $x(t)$ be the concentration of the microbial *Clostridium Acetobutylicum* P262 in batch fermentation process at time t (measured in hours). If the average of birth rate μ_{max} is constant and the average of death rate κ is proportional to the cell mass, then the cell growth concentration rate of *Clostridium Acetobutylicum* P262 can be expressed by logistic equation

$$dx(t) = \Big(\mu_{max} - \kappa\Big)x(t)dt$$

where $\kappa = \frac{\mu_{max}}{\eta_{max}}x(t)$ and η_{max} is a carrying capacity of the microbe. Suppose $x(0) = 0.5$ g/L, $\mu_{max} = 0.2$ and $\eta_{max} = 2.88$. Find the concentration of *Clostridium Acetobutylicum* P262 in fermentation process after 3 hours by using a Euler method. Use a step size of 0.5. ■

RK2 Method

Exercise 9.4 Estimate the initial value problems of Exercise 9.1 in subsection 9.1 by using:

i. RK2 of Heun method

ii. RK2 of midpoint method

iii. RK2 of Ralston method

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Exercise 9.5 An inductor and a linear resistor of resistance are connected in series with a DC power source and a switch. The switch is closed at initial time $t_0 = 0$. The mathematical equation of the current I of the circuit is

$$2\frac{dI}{dt} + I = 50$$

Use a midpoint method with a time step of 0.25 s and zero initial current to determine the current at $t = 1.0$ s. ■

Exercise 9.6 A spherical water tank of a radius $R_1 = 15$ m is emptied through a small circular hole of $R_2 = 0.5$ m. The instantaneous water level, H in the tank that is measured from the bottom of the tank can be determined by numerically integrate the following ODE

$$\frac{dH}{dt} = -\frac{5R_2^2\sqrt{2gH}}{2HR_1 - H^2}$$

The term $g = 9.81$ m/s^2 represent the gravity force. The water level $H = 9$ m at initial time $t_0 = 0$ s. Determine the level of water in the tank after $t = 9.0$ s by using RK2 of Ralston method with a time step of 0.5 s. ■

Exercise 9.7 Determine the concentration of *Clostridium Acetobutylicum* P262 in Exercise 9.3 of subsection 9.1 by using Heun method. ■

RK4 Method

Exercise 9.8 Use RK4 method with a step size of 0.5 to approximate the solutions of the following initial value problems.

i. $y' = yt - \sin^2 y, \qquad 0 \le t \le 2, \quad y(0) = 1$

ii. $y' = 1 + \frac{y}{2t^2}$, $\quad 1 \le t \le 2, \quad y(1) = 2$
iii. $\frac{y'}{\sqrt{y}} = 1 + 2xy$, $\quad 0 \le x \le 1, \quad y(0) = 0.5$
iv. $y' = x(y - x^3)$, $\quad 0 \le x \le 3, \quad y(0) = 1$
v. $y' = \ln(2y) - x$, $\quad 0 \le x \le 2, \quad y(0) = 2$

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Exercise 9.9 The rate of heat flow between two points on a heated cylinder at one end is given by

$$\frac{dQ}{dt} = \lambda A \left(\frac{100(L-x)(20-t)}{100-xt} \right)$$

where $\lambda = 0.4$ cal $\cdot$ cm/s is a constant, $A = 10$ cm represent the cylinder's cross–sectional area, $L = 20$ cm is the length of the rod, $x = 2.5$ cm is the distance from the heated end and $Q(0) = 0$ is the initial condition of heat flow at $t_0 = 0$. Compute the heat flow for $0 \le t \le 9$ by using fourth order Runge–Kutta method with a step size of 3. ■

Exercise 9.10 Find the concentration of *Clostridium Acetobutylicum* P262 in Exercise 9.3 of subsection 9.1 by using RK4 method. ■

System of ODEs

Exercise 9.11 Solve the system

$$\begin{aligned} \frac{dy}{dx} &= \sin(x) + \cos(y) + \sin(z) \\ \frac{dz}{dx} &= \cos(x) + \sin(z) \end{aligned}$$

with initial condition of $y(0) = 2.5689$ and $z(0) = 1.5689$ over the interval $0 \le x \le 2$ using
i. Euler method
ii. Fourth order Runge–Kutta method
Use a step size of 0.5. ■

Exercise 9.12 Estimate the solution of the system

$$\begin{aligned} x' &= 2\exp(x) + 3\sin(y), \qquad x(0) = 0.1252 \\ y' &= \cos(xy), \qquad y(0) = 0.9234 \end{aligned}$$

over the interval $0 \le t \le 1$ for a step size of 0.5 by using
i. Euler method
ii. Fourth order Runge–Kutta method

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Exercise 9.13 In the Lotka–Volterra model, under the assumption that the prey, x, learn to avoid the predators, y, the growth and decay rates due to predation will depend on the independent variable, t can be represented as

$$\frac{dx}{dt} = ax - bxy, \qquad x(0) = 4$$
$$\frac{dy}{dt} = -cy + rxy, \qquad y(0) = 2$$

where $a = 0.8, b = 0.5, c = 0.5$ and $r = 2.0$. Let a step size $\Delta t = 0.5$, find $x(2)$ and $y(2)$ using Euler method. ■

Exercise 9.14 Two cylindrical water tanks are connected. Initially there are 10 litres of water in the top tank and 5 litres in the bottom tank. The valve between the two tanks are opened at initial time, $t_0 = 0$. The flow rate through each of these valves is proportional to the volume of the water in the tank. Volume of water in both tanks (v_1–volume of the water in the top tank, v_2–volume of water in the bottom tank) can be described by a system of two first order ODEs

$$\frac{dv_1}{dt} = -0.8v_1$$
$$\frac{dv_2}{dt} = 0.1v_1 + \cosh t + 2.5v_2$$

Numerically integrate $v_1(3)$ and $v_2(3)$ by using RK4 method. Use a step size of 0.5. ■

References 1. Chapra, C. S. & Canale, R. P. Numerical Methods for Engineers, Sixth Edition, McGraw–Hill, 2010.