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Numerical Methods
Ordinary Differential Equations: Initial Value Problems (IVP)

By

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Description

AIMS

This chapter is aimed to solve initial value problems of single ODE by using three different types of methods involving Euler's method, 2nd order Runge-Kutta method and 4th order Runge-Kutta method. In addition, for system of ODEs, two types of methods are considered; Euler's method and 4th order Runge-Kutta method. Steps by steps of solving initial value problems for single ODE and system of ODEs are presented

EXPECTED OUTCOMES

1. Students should be able to solve initial value problems using Euler's method, 2nd order Runge-Kutta method and 4th order Runge-Kutta method.
2. Students should be able to solve system of ODEs using Euler's method and 4th order Runge-Kutta method.

REFERENCES

1. Norhayati Rosli, Nadirah Mohd Nasir, Mohd Zuki Salleh, Rozieana Khairuddin, Nurfatihah Mohamad Hanafi, Noraziah Adzhar. *Numerical Methods,* Second Edition, UMP, 2017 (Internal use)
2. Chapra, C. S. & Canale, R. P. *Numerical Methods for Engineers*, Sixth Edition, McGraw–Hill, 2010.

Content

INTRODUCTION

- ODEs refers as a rate of equation .
- It expresses the rate of change of a variable as a function of variables and parameters .
- ODEs provide a tool for better understanding the behaviour of many biological and physical systems around us.
- It forms the basis of simulation experiments in a realm where the experiments
  are often impractical or unethical.
- By solving the underlying ODEs, one can identify trends and make forecasts about the future path of the process.

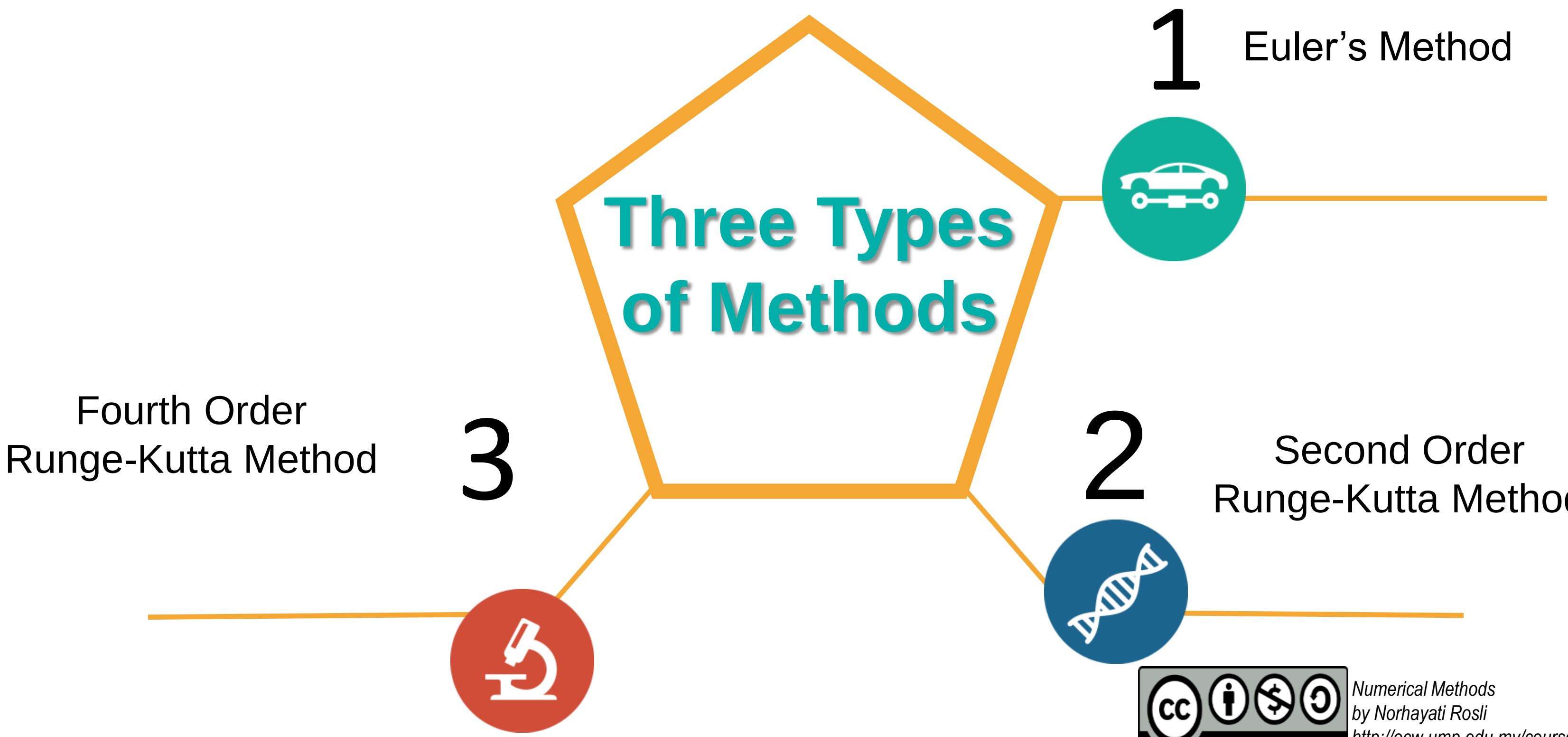
General Form of ODEs

$$\frac{dy}{dx} = f(x, y)$$

INTRODUCTION (Cont.)

- Most of ODEs cannot be solved analytically.
- Its due to the complexity form of the equations.
- Numerical methods offer a viable option to solve ODEs.

Numerical Methods for Solving ODEs (IVP)



EULER'S METHOD

- Euler's method is a one step method and can be formulated in general as

$$y_{i+1} = y_i + \Phi h$$

where h denotes a step size and Φ is a slope estimate.

- A new value of y_{i+1} is extrapolated from an old value of y_i over a distance, h.
- For Euler's method the first derivative $\frac{dy}{dx}|_{x_i} = f(x_i, y_i)$ provide the slope estimate at x_i such that

$$\Phi = f\left(x_i, y_i\right)$$

EULER's METHOD (Cont.)

Euler's Method Formula

$$y_{i+1} = y_i + f\left(x_i, y_i\right)h$$

$$x_{i+1} = x_i + h$$

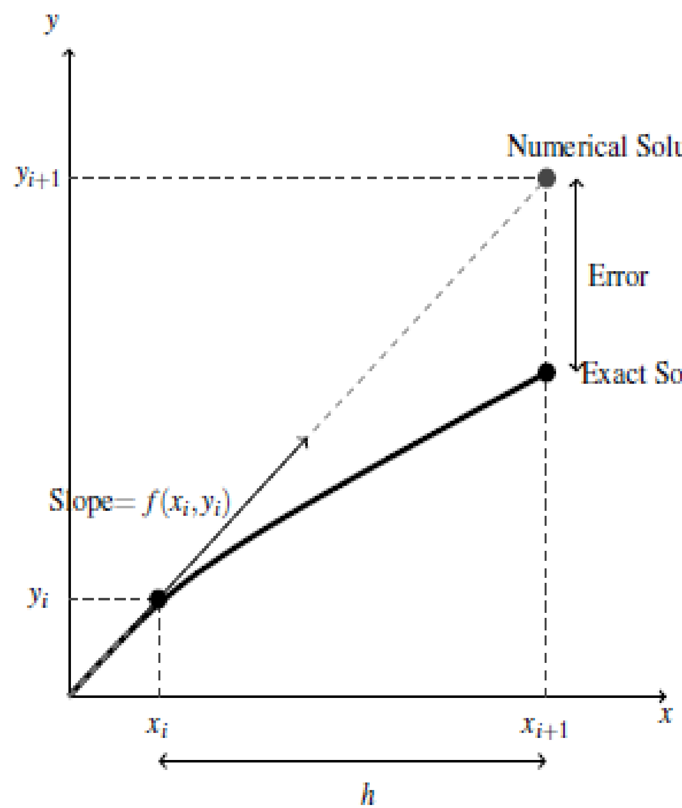


Figure 1: Graphical Illustration of Euler's Method

EULER's METHOD (Cont.)

Example 1

Use Euler's method to solve the following ODE (IVP)

$$\frac{dy}{dx} = \exp(-x) - 2y,\ y(0) = 2$$

for $0 \le x \le 2$ with a step size, $h = 0.5$.

Solution

Step 1 Identify the estimate slope, $f(x, y) = \exp(-x) - 2y$, initial values, $x_0 = 0, y_0 = 2$ and $h = 0.5$.

Step 2 Approximate iteratively $y_{i+1} = y(x_{i+1})$ over the interval $0 \le x \le 2$ by using Euler's method.

EULER's METHOD (Cont.)

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Solution (Cont.)

First iteration: $i = 0, x_0 = 0, y_0 = 2$

$$y_1 = y_0 + f(x_0, y_0)(0.5)$$
$$= 2 + f(0,2)(0.5)$$
$$= 2 + (\exp(-0) - 2(2))(0.5)$$
$$= 0.5$$
$$x_1 = x_0 + h$$
$$= 0 + 0.5 = 0.5$$
$$y_1 \approx y(0.5) = 0.5$$

Second iteration: $i = 1, x_1 = 0.5, y_1 = 0.5$

$$y_2 = y_1 + f(x_1, y_1)(0.5)$$
$$= 0.5 + f(0.5, 0.5)(0.5)$$
$$= 0.3033$$
$$x_2 = x_1 + h$$
$$= 1.0$$
$$y_2 \approx y(1.0) = 0.3033$$

EULER's METHOD (Cont.)

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Solution (Cont.)

Third iteration: $i = 2, x_2 = 1.0, y_2 = 0.3033$

$$y_3 = y_2 + f\left(x_2, y_2\right)(0.5)$$
$$= 0.3033 + f(1.0, 0.3033)(0.5)$$
$$= 0.1839$$
$$x_3 = 1.5$$
$$y_3 \approx y(1.5) = 0.1839$$

Fourth iteration: $i = 3, x_3 = 1.5, y_3 = 0.1839$

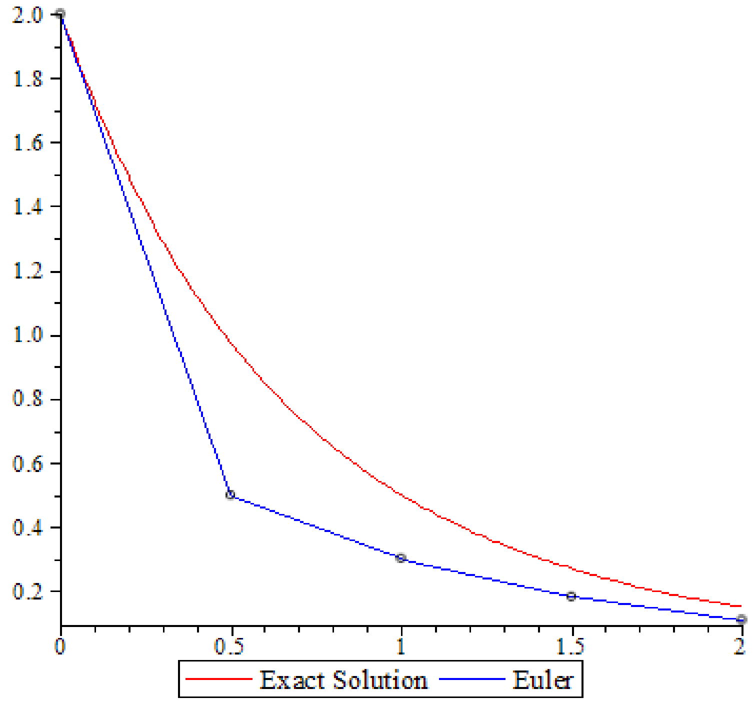
$$y_4 = y_3 + f\left(x_3, y_3\right)(0.5)$$
$$= 0.1839 + f(1.5, 0.1839)(0.5)$$
$$= 0.1116$$
$$x_4 = 2.0$$
$$y_4 \approx y(2.0) = 0.1116$$

EULER's METHOD (Cont.)

Solution (Cont.)

The solution is summarised in the following table and figure. Figure below shows the comparison of the approximate solutions for Example 1 and the exact solutions.

i	x_i	y_i
0	0	2
1	0.5	0.5
2	1.0	0.3033
3	1.5	0.1839
4	2.0	0.1116



SECOND ORDER RUNGE-KUTTA METHODS

General Form of Second Order Runge-Kutta Methods

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

where

$$a_1 + a_2 = 1$$

$$a_2 p_1 = \frac{1}{2}$$

$$a_2 q_{11} = \frac{1}{2}$$

General Formula of RK2 Methods

SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

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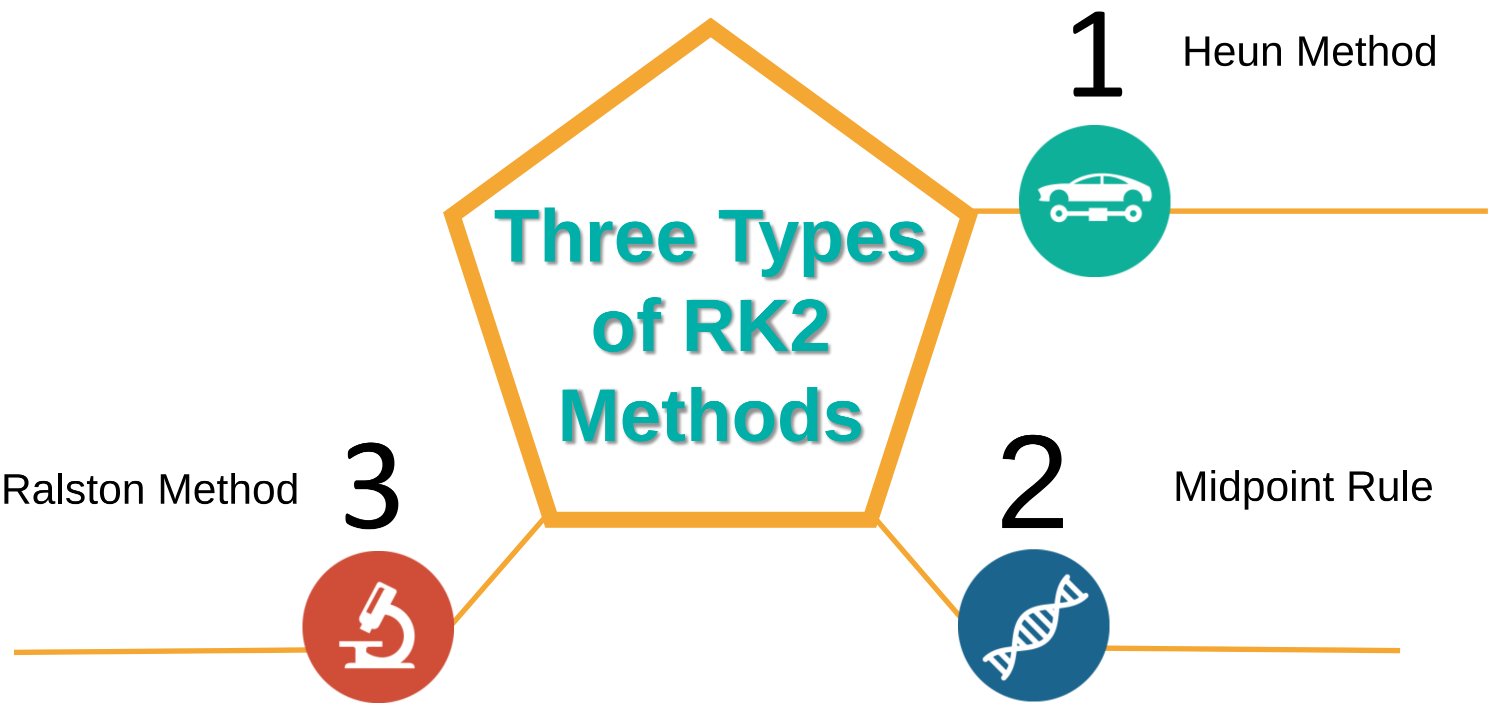
Based on the General Form of RK2, Three Types of Methods are Developed.



SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

Heun Method with a Single Corrector ($a_2 = 1/2$)

Heun Method Formula

$$y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)h$$

$$x_{i+1} = x_i + h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

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Midpoint Method ($a_2 = 1$)

Midpoint Method Formula

$$y_{i+1} = y_i + k_2 h$$

$$x_{i+1} = x_i + h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

Ralston Method ($a_2 = 2/3$)

Ralston Method Formula

$$y_{i+1} = y_i + \frac{1}{3}(k_1 + 2k_2)h$$

$$x_{i+1} = x_i + h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h)$$

SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

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Example 2

Use RK2 of Heun method to solve the following ODE (IVP)

$$\frac{dy}{dx} = \exp(-x) - 2y, \; y(0) = 2$$

for $0 \leq x \leq 2$ with a step size, $h = 0.5$.

Solution

Step 1

Identify the estimate slope, $f(x, y) = \exp(-x) - 2y$, initial values, $x_0 = 0, y_0 = 2$ and $h = 0.5$.

Step 2

Approximate iteratively $y_{i+1} = y(x_{i+1})$ over the interval $0 \leq x \leq 2$ by using Heun method.

SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

Solution (Cont.)

First iteration: $i=0, x_0=0, y_0=2$

$$k_1 = f(x_0, y_0) = f(0,2) = -3$$

$$k_2 = f(x_0 + h, y_0 + k_1 h)$$
$$= f(0.5, 0.5) = -0.3935$$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)(h)$$
$$= 2 + \frac{1}{2}(-3 + (-0.3935))(0.5)$$
$$= 1.1516$$

$$x_1 = x_0 + h$$
$$= 0 + 0.5 = 0.5$$

$$y_1 \approx y(0.5) = 1.1516$$

Second iteration: $i=1, x_1=0.5, y_1=1.1516$

$$k_1 = f(0.5, 1.516) = -1.6967$$

$$k_2 = f(1.0, 0.3033) = -0.2387$$

$$y_2 = 0.6678$$

$$x_2 = 1.0$$

$$y_1 \approx y(1.0) = 0.6678$$

SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

Solution (Cont.)

Third iteration: $i = 2, x_2 = 1.0, y_2 = 0.6678$

$$k_1 = f(1.0, 0.6678) = -0.9677$$

$$k_2 = f(1.5, 0.1839) = -0.1447$$

$$y_3 = 0.3897$$

$$x_3 = 1.5$$

$$y_3 \approx y(1.5) = 0.3897$$

Fourth iteration: $i = 3, x_3 = 1.5, y_3 = 0.3897$

$$k_1 = f(1.0, 0.3897) = -0.5562$$

$$k_2 = f(2.0, 0.1116) = -0.0878$$

$$y_4 = 0.2287$$

$$x_4 = 2.0$$

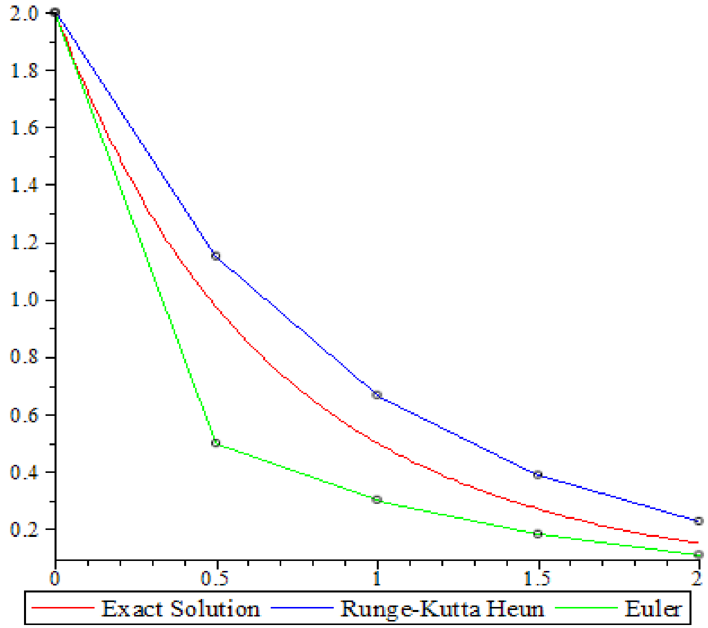
$$y_4 \approx y(2.0) = 0.2287$$

SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

Solution (Cont.)

The solution is summarised in the following table and figure. Figure shows the comparison of the approximate solutions using Heun method, Euler method and the exact solutions.

i	x_i	y_i
0	0	2
1	0.5	1.1516
2	1.0	0.6678
3	1.5	0.3897
4	2.0	0.2287



SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

Example 3

Use RK2 of Midpoint method to solve the following ODE (IVP)

$$\frac{dy}{dx} = \exp(-x) - 2y, \; y(0) = 2$$

for $0 \le x \le 2$ with a step size, $h = 0.5$.

Solution

Step 1

Identify the estimate slope, $f(x,y) = \exp(-x) - 2y$, initial values, $x_0 = 0, y_0 = 2$ and $h = 0.5$.

Step 2

Approximate iteratively $y_{i+1} = y(x_{i+1})$ over the interval $0 \le x \le 2$ by using midpoint method.

SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

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Solution (Cont.)

First iteration: $i = 0, x_0 = 0, y_0 = 2$

$$k_1 = f\left(x_0, y_0\right) = f(0,2) = -3$$

$$k_2 = f\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1 h\right)$$

$$= f(0.25, 1.25) = -1.7212$$

$$y_1 = y_0 + k_2(h)$$

$$= 2 + (-1.7212)(0.5)$$

$$= 1.1394$$

$$x_1 = x_0 + h$$

$$= 0 + 0.5 = 0.5$$

$$y_1 \approx y(0.5) = 1.1394$$

Second iteration: $i = 1, x_1 = 0.5, y_1 = 1.1394$

$$k_1 = f(0.5, 1.1394) = -1.6723$$

$$k_2 = f(0.75, 0.7213) = -0.9703$$

$$y_2 = 0.6543$$

$$x_2 = 1.0$$

$$y_1 \approx y(1.0) = 0.6543$$

SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

Solution (Cont.)

Third iteration: $i = 2, x_2 = 1.0, y_2 = 0.6543$

$$k_1 = f(1.0, 0.6543) = -0.9407$$

$$k_2 = f(1.25, 0.4191) = -0.5517$$

$$y_3 = 0.3784$$

$$x_3 = 1.5$$

$$y_3 \approx y(1.5) = 0.3784$$

Fourth iteration: $i = 3, x_3 = 1.5, y_3 = 0.3784$

$$k_1 = f(1.5, 0.3784) = -0.5337$$

$$k_2 = f(1.75, 0.2450) = -0.3162$$

$$y_4 = 0.2203$$

$$x_4 = 2.0$$

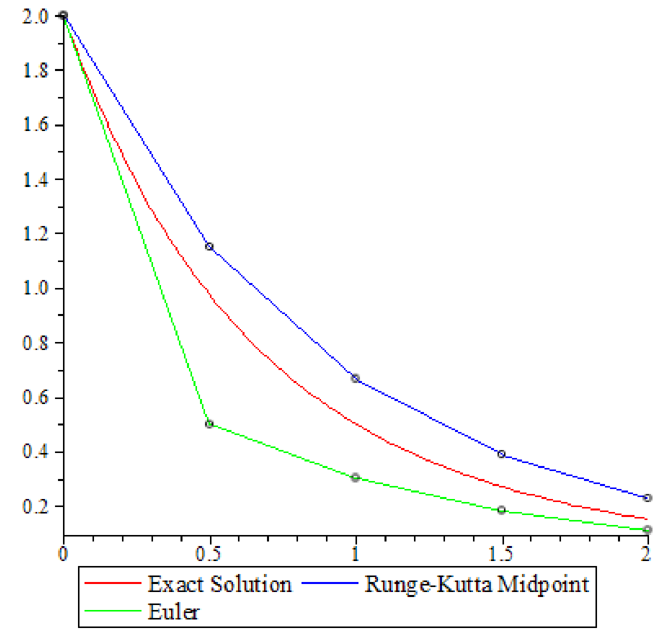
$$y_4 \approx y(2.0) = 0.2203$$

SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

Solution (Cont.)

The solution is summarised in the following table and figure. Figure below shows the comparison of the approximate solutions using midpoint method, Euler method and the exact solutions.

i	x_i	y_i
0	0	2
1	0.5	1.1394
2	1.0	0.6543
3	1.5	0.3784
4	2.0	0.2203



SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

Example 4

Use RK2 of Ralston method to solve the following ODE (IVP)

$$\frac{dy}{dx} = \exp(-x) - 2y, \; y(0) = 2$$

for $0 \leq x \leq 2$ with a step size, $h = 0.5$.

Solution

Step 1 Identify the estimate slope, $f(x, y) = \exp(-x) - 2y$, initial values, $x_0 = 0, y_0 = 2$ and $h = 0.5$.

Step 2 Approximate iteratively $y_{i+1} = y(x_{i+1})$ over the interval $0 \leq x \leq 2$ by using Ralston method.

SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

Solution (Cont.)

First iteration: $i = 0, x_0 = 0, y_0 = 2$

$$k_1 = f\left(x_0, y_0\right) = f(0,2) = -3$$

$$k_2 = f\left(x_0 + \tfrac{3}{4}h, y_0 + \tfrac{3}{4}k_1 h\right)$$

$$= f(0.375, 0.8755) = -1.0627$$

$$y_1 = y_0 + \frac{1}{3}(k_1 + 2k_2)(h)$$

$$= 2 + \frac{1}{3}(-3 + 2(-1.0627))(0.5)$$

$$= 1.1458$$

$$x_1 = x_0 + h$$

$$= 0 + 0.5 = 0.5$$

$$y_1 \approx y(0.5) = 1.1458$$

Second iteration: $i = 1, x_1 = 0.5, y_1 = 1.1458$

$$k_1 = f(0.5, 1.1458) = -1.6850$$

$$k_2 = f(0.875, 0.5139) = -0.6109$$

$$y_2 = 0.6613$$

$$x_2 = 1.0$$

$$y_1 \approx y(1.0) = 0.6613$$

SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

Solution (Cont.)

Third iteration: $i = 2, x_2 = 1.0, y_2 = 0.6613$

$$k_1 = f(1.0, 0.6613) = -0.9547$$
$$k_2 = f(1.375, 0.3033) = -0.3537$$
$$y_3 = 0.3843$$
$$x_3 = 1.5$$
$$y_3 \approx y(1.5) = 0.3843$$

Fourth iteration: $i = 3, x_3 = 1.5, y_3 = 0.3843$

$$k_1 = f(1.5, 0.3843) = -0.5454$$
$$k_2 = f(1.875, 0.1797) = -0.2061$$
$$y_4 = 0.2247$$
$$x_4 = 2.0$$
$$y_4 \approx y(2.0) = 0.2247$$

SECOND ORDER RUNGE-KUTTA METHODS (Cont.)

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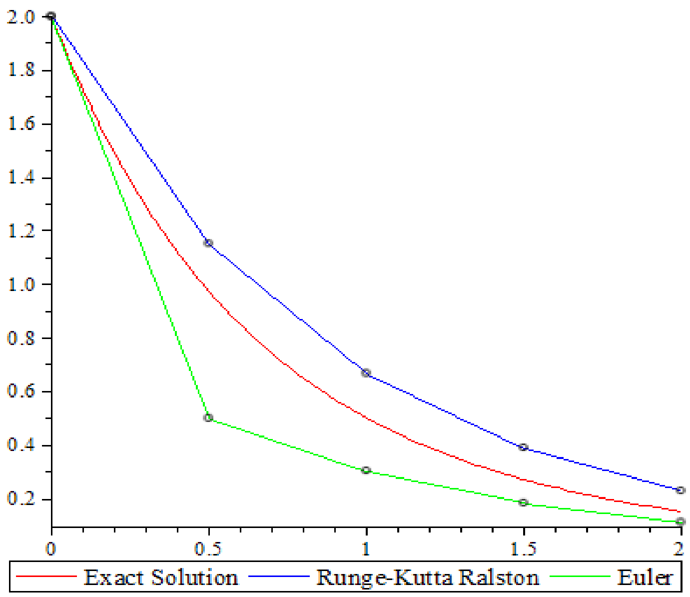
Solution (Cont.)

The solution is summarised in the following table and figure. Figure below shows the comparison of the approximate solutions using Ralston method, Euler method and the exact solutions.

i	x_i	y_i
0	0	2
1	0.5	1.1458
2	1.0	0.6613
3	1.5	0.3843
4	2.0	0.2247



FOURTH ORDER RUNGE-KUTTA METHOD

- The most popular Runge-Kutta method is often referred to as fourth order Runge-Kutta (RK4).
- It was developed around 1900 by the German Mathematicians C. Runge and M. W. Kutta.
- RK4 is normally known as classical fourth–order RK method.

Fourth Order Runge-Kutta Method Formula

The next value of $y(x_{i+1})$ is determined by the sum of the current value of $y(x_i)$ and the weighted average of four increments. The terms k's represent:

- k_1 is the increment of the slope at the beginning of the interval, using y
- k_2 is the increment of the slope at the midpoint of the interval, using k_1
- k_3 is the increment of the slope at the midpoint of the interval, using k_2
- k_4 is the increment of the slope at the end of the interval, using k_3

FOURTH ORDER RUNGE-KUTTA METHOD (Cont.)

Fourth Order Runge-Kutta Method Formula

Fourth Order Runge-Kutta Method Formula

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$x_{i+1} = x_i + h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f\left(x_i + h, y_i + k_3 h\right)$$

FOURTH ORDER RUNGE-KUTTA METHODS (Cont.)

Example 5

Use RK4 method to solve the following ODE (IVP)

$$\frac{dy}{dx} = \exp(-x) - 2y,\ y(0) = 2$$

for $0 \le x \le 2$ with a step size, $h = 0.5$.

Solution

Step 1

Identify the estimate slope, $f(x, y) = \exp(-x) - 2y$, initial values, $x_0 = 0, y_0 = 2$ and $h = 0.5$.

Step 2

Approximate iteratively $y_{i+1} = y(x_{i+1})$ over the interval $0 \le x \le 2$ by using RK4 method.

FOURTH ORDER RUNGE-KUTTA METHODS (Cont.)

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Solution (Cont.)

First iteration: $i = 0, x_0 = 0, y_0 = 2$

$$k_1 = f(0,2) = -3$$

$$k_2 = f\left(0 + \tfrac{1}{2}(0.5), 2 + \tfrac{1}{2}(-3)(0.5)\right)$$
$$= f(0.25, 1.25) = -1.7212$$

$$k_3 = f\left(0 + \tfrac{1}{2}(0.5), 2 + \tfrac{1}{2}(-1.7212)(0.5)\right)$$
$$= f(0.25, 1.5697) = -2.3606$$

$$k_4 = f\left(0.5, 2 + (-2.3606)(0.5)\right)$$
$$= f(0.5, 0.8197) = -1.0329$$

$$y_1 = 2 + \frac{0.5}{6}(-3 + 2(-1.7212) + 2(-2.3606) + (-1.0329))$$
$$= 0.9836$$

$$x_1 = 0.5$$

$$y_1 \approx y(0.5) = 0.9836$$

FOURTH ORDER RUNGE-KUTTA METHODS (Cont.)

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Solution (Cont.)

Second iteration: $i = 1, x_1 = 0.5, y_1 = 0.9836$

$$k_1 = f(0.5, 0.9836) = -1.3607$$
$$k_2 = f(0.75, 0.6434) = -0.8145$$
$$k_3 = f(0.75, 0.7800) = -1.0876$$
$$k_4 = f(1.0, 0.4398) = -0.5118$$
$$y_2 = 0.5106$$
$$x_2 = 1.0$$
$$y_2 \approx y(1.0) = 0.5106$$

Third iteration: $i = 2, x_2 = 1.0, y_2 = 0.5106$

$$k_1 = f(1.0, 0.5106) = -0.6532$$
$$k_2 = f(1.25, 0.3473) = -0.4080$$
$$k_3 = f(1.25, 0.4086) = -0.5306$$
$$k_4 = f(1.5, 0.2453) = -0.2674$$
$$y_3 = 0.2774$$
$$x_3 = 1.5$$
$$y_3 \approx y(1.5) = 0.2774$$

FOURTH ORDER RUNGE-KUTTA METHODS (Cont.)

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Solution (Cont.)

Fourth iteration: $i = 3, x_3 = 1.5, y_3 = 0.1839$

$$k_1 = f(1.5, 0.2774) = -0.3317$$

$$k_2 = f(1.75, 0.6434) = -0.2152$$

$$k_3 = f(1.75, 0.7800) = -0.2734$$

$$k_4 = f(2.0, 0.4398) = -0.1460$$

$$y_4 = 0.1562$$

$$x_4 = 2.0$$

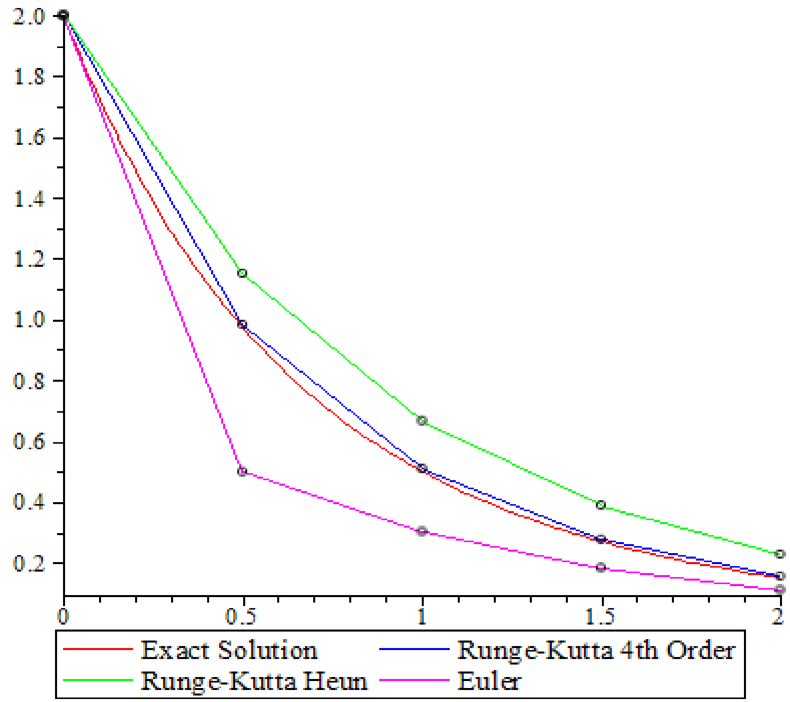
$$y_4 \approx y(2.0) = 0.1562$$

FOURTH ORDER RUNGE-KUTTA METHODS (Cont.)

Solution (Cont.)

The solution is summarised in the following table and figure. Figure below shows the comparison of the approximate solutions using RK4 method, Heun method, Euler method and the exact solutions.

i	x_i	y_i
0	0	2
1	0.5	0.9836
2	1.0	0.5106
3	1.5	0.2774
4	2.0	0.1562



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SYSTEM OF ODEs

- Many practical problems in science and engineering need to be modelled in the form of a system of ODEs rather than single ODE.
- In general, such system can be represented as

$$\frac{dy_1}{dx} = f_1(x, y_1, \ldots, y_n)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, \ldots, y_n)$$

$$\vdots$$

$$\frac{dy_n}{dx} = f_n(x, y_1, \ldots, y_n)$$

which requires n initial conditions at the starting values of x.

SYSTEM OF ODEs (Cont.)

System of ODEs with $n = 2$

In general, a system of two first order ODEs with y and z referred to as dependent variables and x referred to as independent variable has the form

$$\frac{dy}{dx} = f_1(x, y, z)$$

$$\frac{dz}{dx} = f_2(x, y, z)$$

for the domain $x_0 \leq x \leq x_n$

with initial condition

$y(x_0) = y_0$ and $z(x_0) = z_0$

SYSTEM OF ODEs (Cont.)

Euler's Method for System of ODEs

Euler's Method Formula

$$x_{i+1} = x_i + h$$

$$y_{i+1} = y_i + f_1(x_i, y_i, z_i)h$$

$$z_{i+1} = z_i + f_2(x_i, y_i, z_i)h$$

SYSTEM OF ODEs (Cont.)

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RK4 Method for System of ODEs

RK4 Method Formula

$$y_{i+1} = y_i + \frac{h}{6}\left(k_{y,1} + 2k_{y,2} + 2k_{y,3} + k_{y,4}\right)$$

$$z_{i+1} = z_i + \frac{h}{6}\left(k_{z,1} + 2k_{z,2} + 2k_{z,3} + k_{z,4}\right)$$

$$x_{i+1} = x_i + h$$

$$k_{y,1} = f_1\left(x_i, y_i, z_i\right), \quad k_{z,1} = f_2\left(x_i, y_i, z_i\right)$$

$$k_{y,2} = f_1\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_{y,1}h, z_i + \frac{1}{2}k_{z,1}h\right)$$

$$k_{z,2} = f_2\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_{y,1}h, z_i + \frac{1}{2}k_{z,1}h\right)$$

$$k_{y,3} = f_1\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_{y,2}h, z_i + \frac{1}{2}k_{z,2}h\right)$$

$$k_{z,3} = f_2\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_{y,2}h, z_i + \frac{1}{2}k_{z,2}h\right)$$

$$k_{y,4} = f_1\left(x_i + h, y_i + k_{y,3}h, z_i + k_{z,3}h\right)$$

$$k_{z,4} = f_2\left(x_i + h, y_i + k_{y,3}h, z_i + k_{z,3}h\right)$$

SYSTEM OF ODEs (Cont.)

Example 6

Use Euler's method to solve the following system of ODEs

$$\frac{dy}{dx} = -2y + 4e^{-x}, \quad y(0) = 2$$

$$\frac{dz}{dx} = -\frac{yz^2}{3}, \quad z(0) = 4$$

for $0 \le x \le 1$ with a step size, $h = 0.5$.

Solution

Step 1

Identify $f_1(x, y, z)$ and $f_2(x, y, z)$

$$\frac{dy}{dx} = f_1(x, y, z) = -2y + 4e^{-x}, \; y(0) = 2$$

$$\frac{dz}{dx} = f_2(x, y, z) = -\frac{yz^2}{3}, \; z(0) = 4$$

Step 2

Approximate iteratively $y_{i+1} = y(x_{i+1})$ and $z_{i+1} = z(x_{i+1})$ over the interval $0 \le x \le 1$ by using Euler's method.

SYSTEM OF ODEs (Cont.)

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Solution (Cont.)

First iteration: $i = 0, x_0 = 0, y_0 = 2, z_0 = 4$

$$y_1 = 2 + f_1(0,2,4)(0.5)$$
$$= 2$$
$$z_1 = 4 + f_2(0,2,4)(0.5)$$
$$= -1.3333$$
$$x_1 = 0.5$$
$$y_1 \approx y(0.5) = 2$$
$$z_1 \approx z(0.5) = -1.3333$$

SYSTEM OF ODEs (Cont.)

Solution (Cont.)

Second iteration: $i = 1, x_1 = 0.5, y_1 = 0.5, z_1 = -1.3333$

$$y_2 = 2 + f_1(0.5, 2, -1.3333)(0.5)$$
$$= 1.2131$$
$$z_2 = -1.3333 + f_2(0.5, 2, -1.3333)(0.5)$$
$$= -1.9259$$
$$x_2 = 1.0$$
$$y_2 \approx y(1.0) = 1.2131$$
$$z_2 \approx z(1.0) = -1.9259$$

SYSTEM OF ODEs (Cont.)

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Solution (Cont.)

The solution is summarised in the following

i	x_i	y_i	z_i
0	0	2	4
1	0.5	2	-1.3333
2	1.0	1.2131	-1.9259

SYSTEM OF ODEs (Cont.)

Example 7

Use RK4 method to solve the following system of ODEs

$$\frac{dy}{dx} = -2y + 4e^{-x}, \quad y(0) = 2$$

$$\frac{dz}{dx} = -\frac{yz^2}{3}, \quad z(0) = 4$$

for $0 \le x \le 1$ with a step size, $h = 0.5$.

Solution

Step 1

Identify $f_1(x, y, z)$ and $f_2(x, y, z)$

$$\frac{dy}{dx} = f_1(x, y, z) = -2y + 4e^{-x}, \; y(0) = 2$$

$$\frac{dz}{dx} = f_2(x, y, z) = -\frac{yz^2}{3}, \; z(0) = 4$$

Step 2

Approximate iteratively $y_{i+1} = y(x_{i+1})$ and $z_{i+1} = z(x_{i+1})$ over the interval $0 \le x \le 1$ by using RK4 method.

SYSTEM OF ODEs (Cont.)

Solution (Cont.)

First iteration: $i = 0, x_0 = 0, y_0 = 2, z_0 = 4$

$$
\begin{aligned}
k_{y,1} &= f_1(x_0, y_0, z_0) \\
&= f_1(0, 2, 4) \\
&= \left[-2(2) + 4e^{-(0)}\right] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
k_{z,1} &= f_2(x_0, y_0, z_0) \\
&= f_2(0, 2, 4) \\
&= \left[\frac{-2(4)^2}{3}\right] \\
&= -10.6667
\end{aligned}
$$

$$
\begin{aligned}
k_{y,2} &= f_1(x_0 + \frac{h}{2}, y_0 + \frac{k_{y,1}}{2}h, z_0 + \frac{k_{z,1}}{2}h) \\
&= f_1(0.25, 2, 1.3334) \\
&= \left[-2(2) + 4e^{-(0.25)}\right] \\
&= -0.8848
\end{aligned}
$$

$$
\begin{aligned}
k_{z,2} &= f_2(x_0 + \frac{h}{2}, y_0 + \frac{k_{y,1}}{2}h, z_0 + \frac{k_{z,1}}{2}h) \\
&= f_2(0.25, 2, 1.3334) \\
&= \left[\frac{-2(1.3334)^2}{3}\right] \\
&= -1.1853
\end{aligned}
$$

SYSTEM OF ODEs (Cont.)

Solution (Cont.)

First iteration: $i = 0, x_0 = 0, y_0 = 2, z_0 = 4$

$$k_{y,3} = f_1(x_0 + \frac{h}{2}, y_0 + \frac{k_{y,2}}{2}h, z_0 + \frac{k_{z,2}}{2}h)$$
$$= f_1(0.25, 1.7788, 3.7037)$$
$$= \left[-2(1.7788) + 4e^{-(0.25)}\right]$$
$$= -0.4424$$

$$k_{z,3} = f_2(x_0 + \frac{h}{2}, y_0 + \frac{k_{y,2}}{2}h, z_0 + \frac{k_{z,2}}{2}h)$$
$$= f_2(0.25, 1.7788, 3.7037)$$
$$= \left[\frac{-(1.7788)(3.7037)^2}{3}\right]$$
$$= -8.1335$$

SYSTEM OF ODEs (Cont.)

Solution (Cont.)

First iteration: $i = 0, x_0 = 0, y_0 = 2, z_0 = 4$

$$
\begin{aligned}
k_{y,4} &= f_1(x_0 + h, y_0 + k_{y,3}h, z_0 + k_{z,3}h) \\
&= f_1(0.5, 1.7788, -0.0667) \\
&= \left[-2(1.7788) + 4e^{-(0.5)}\right] \\
&= -1.1315
\end{aligned}
$$

$$
\begin{aligned}
k_{z,4} &= f_2(x_0 + h, y_0 + k_{y,3}h, z_0 + k_{z,3}h) \\
&= f_2(0.5, 1.7788, -0.0667) \\
&= \left[\frac{-(1.7788)(-0.0667)^2}{3}\right] \\
&= 2.6379 \times 10^{-3}
\end{aligned}
$$

SYSTEM OF ODEs (Cont.)

Solution (Cont.)

First iteration: $i = 0, x_0 = 0, y_0 = 2, z_0 = 4$

$$y_1 = y_0 + \frac{h}{6}(k_{y,1} + 2k_{y,2} + 2k_{y,3} + k_{y,4})$$

$$= 2 + \frac{0.5}{6}\left(0 + 2(-0.8848 - 0.4424) - 1.1315\right)$$

$$= 1.6845$$

$$y(0.5) \approx y_1 = 1.6845 \quad x_1 = x_0 + 0.5 = 0.5$$

$$z_1 = z_0 + \frac{h}{6}(k_{z,1} + 2k_{z,2} + 2k_{z,3} + k_{z,4})$$

$$= 4 + \frac{0.5}{6}\left(-10.6667 + 2(-1.1853 - 8.1335) - 2.6379 \times 10^{-3}\right)$$

$$= 1.5578$$

$$z(0.5) \approx z_1 = 1.5578 \quad x_1 = x_0 + 0.5 = 0.5$$

Repeat the process for $i = 1$

Conclusion

- RK4 method has better order of convergence than Euler and RK2 methods.
- However, the main computational effort in applying the RK4 method is one needs to evaluate four functional evaluations per step.
- For instance, in comparing with RK2 method, RK4 method requires twice as many evaluations per step.
- The approximation solution that is obtained by using RK4 method will provide better approximate solution than Euler and RK2 methods.

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