Numerical Methods
Numerical Integrations

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Description

AIMS

This chapter is aimed to solve the integration of the given functions by using numerical integration methods.

EXPECTED OUTCOMES

1. Students should be able to numerically integrate the integration by using Trapezoidal rule and Simpson's rule

REFERENCES

1. Norhayati Rosli, Nadirah Mohd Nasir, Mohd Zuki Salleh, Rozieana Khairuddin, Nurfatihah Mohamad Hanafi, Noraziah Adzhar. *Numerical Methods,* Second Edition, UMP, 2017 (Internal use)
2. Chapra, C. S. & Canale, R. P. *Numerical Methods for Engineers*, Sixth Edition, McGraw–Hill, 2010.

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INTRODUCTION

- Integration is a process of measuring the area under a function $f(x)$ which is plotted on a graph
- Graphical illustration of measuring the area under the curve is depicted in **Figure 1.**

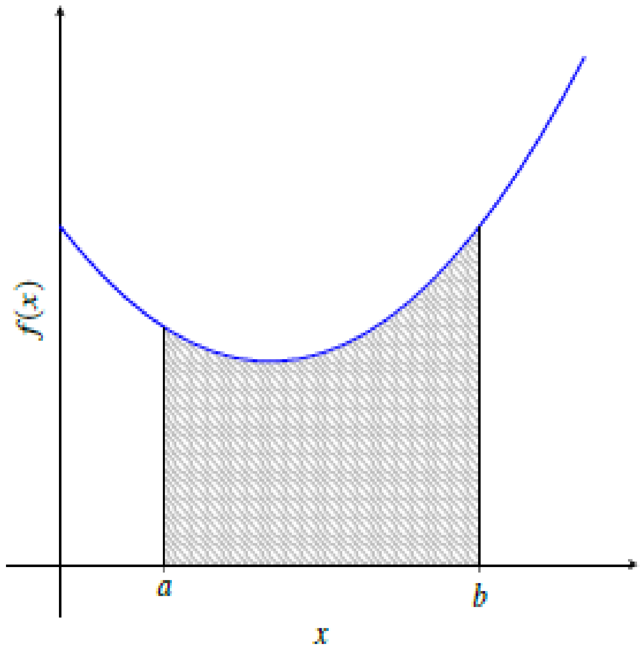


Figure 1: Graphical illustration of measuring the area under the curve

INTRODUCTION (Cont.)

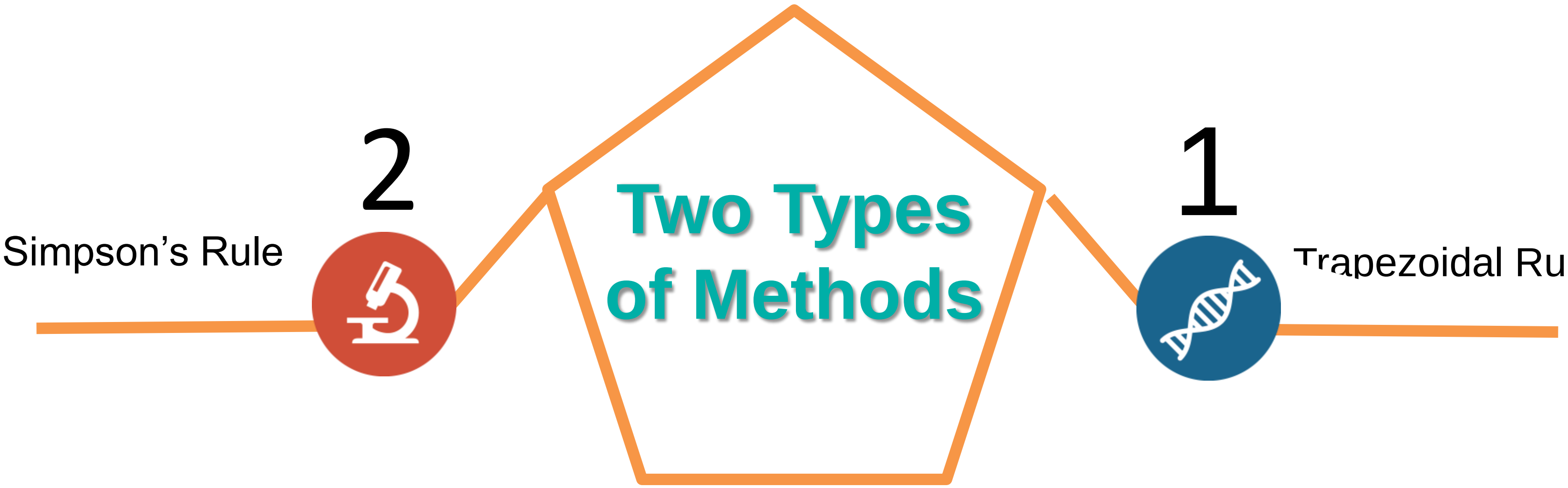
Mathematically, the integration of $f(x)$ can be formulated as

$$I = \int_a^b f(x)dx \qquad (1)$$

- Equation (1) represents the integral of the function $f(x)$ with respect to the independent variable x, that is evaluated between the limits of $x = a$ to $x = b$.
- Integration (1) in most of the cases cannot be solved analytically due to the complexity form of the function, $f(x)$.
- Thus requires **numerical integration methods** to solve the integration (1).

INTRODUCTION (Cont.)

Numerical Integration Methods



NUMERICAL INTEGRATION METHODS

Trapezoidal Rule

- Trapezoidal rule is based on **approximating the integrand,** $f(x)$ by a **first order polynomial.**

- Geometrically, it is equivalent to approximate the area of the trapezoid under the straight line connecting $f(a)$ and $f(b)$ as indicated in **Figure 2.**

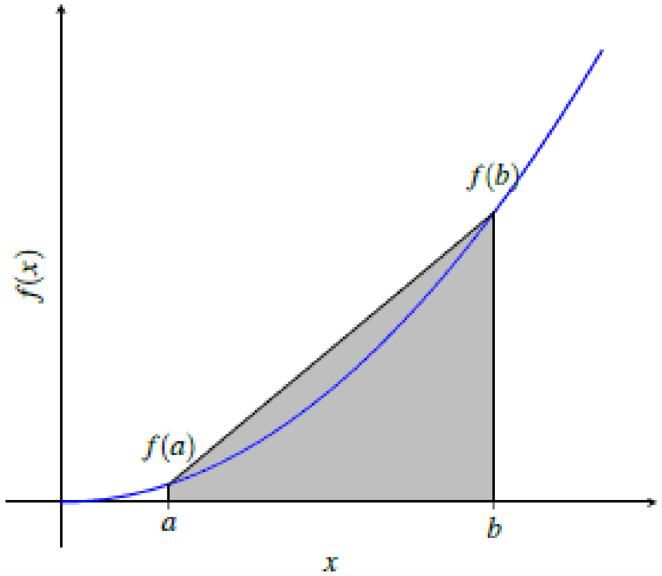


Figure 2: Graphical illustration of the approximation of the integral via Trapezoidal rule

NUMERICAL INTEGRATION METHODS (Cont.)

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Single Trapezoidal rule: Formula

From **Figure 2**, the estimated integral of equation (1) can be represented as

$$I \cong \text{width} \times \text{average height}$$

$$= (b-a) \times \left(\frac{f(a) + f(b)}{2} \right) \qquad (2)$$

Equation (2) can be written as:

$$I = \frac{h}{2}\left(f(a) + f(b) \right)$$

Single application of Trapezoidal rule

NUMERICAL INTEGRATION METHODS (Cont.)

Example 1

Evaluate

$$\int_0^1 \left(\sqrt{\sin^3(x)+1}\right)dx$$

by using Trapezoidal rule.

Solution

$$b=1,\ a=0 \text{ and } h=b-a=1-0=1.$$

$$\int_0^1 \left(\sqrt{\sin^3(x)+1}\right)dx \cong \frac{1-0}{2}\left[f(0)+f(1)\right]$$

$$=0.5(1+1.2633)$$

$$=1.1317$$

Therefore, the numerical integration of Example 1 is 1.1317

NUMERICAL INTEGRATION METHODS (Cont.)

Composite Trapezoidal Rule

- The accuracy of single application of Trapezoidal rule can be improved by dividing the interval $[a, b]$ into a number of finer segments.
- The integral for the entire intervals is computed by adding the areas of the individual segment.
- The method is known as **Composite Trapezoidal rule.**
- The method is developed based on **first order polynomial** by dividing the number of segments, n into equally step size, h.

NUMERICAL INTEGRATION METHODS (Cont.)

Graphical Representation of Composite Trapezoidal Rule

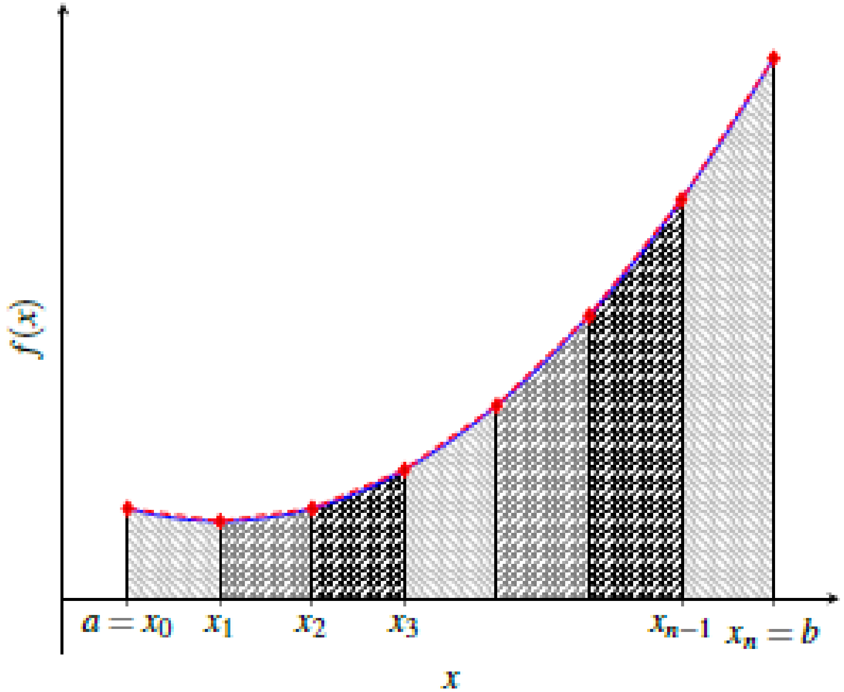

Figure 3: Graphical representation of the approximation of the integral by Composite Trapezoidal rule

NUMERICAL INTEGRATION METHODS (Cont.)

Composite Trapezoidal rule: Formula

Suppose we have $n+1$ equally spaced points, $x_0, x_1, x_2, \cdots, x_n$ with n number of strips of equal width. The step size, h is computed as

$$h = \frac{b-a}{n} = \frac{x_n - x_0}{n}$$

The integration of equation (1) is written as

$$I = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx \qquad (3)$$

NUMERICAL INTEGRATION METHODS (Cont.)

Composite Trapezoidal rule: Formula

Substituting the single application trapezoidal rule into (3) yields

$$I \cong h\left(\frac{f(x_0)+f(x_1)}{2}\right)+h\left(\frac{f(x_1)+f(x_2)}{2}\right)+\cdots+h\left(\frac{f(x_{n-1})+f(x_n)}{2}\right) \quad (4)$$

Grouping the terms of equation (4) gives

$$I=\frac{h}{2}\left[f(x_0)+f\left(x_n)+2\sum_{i=1}^{n-1}f(x_i)\right)\right]$$

Composite Trapezoidal Rule Formula

NUMERICAL INTEGRATION METHODS (Cont.)

Example 2

Evaluate

$$\int_0^1 \left(\sqrt{\sin^3(x)+1}\right)dx$$

by using trapezoidal rule with $n = 10$.

Solution

Composite trapezoidal rule is used, since the number of strips , $n > 1$.

Step 1

$$b = 1,\ a = 0,\ n = 10$$

$$h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$$

Step size, h

NUMERICAL INTEGRATION METHODS (Cont.)

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Solution (Cont.)

For each value of x, find $f(x)$

Step 2

x	$f(x)$
0	1
0.1	1.0005
0.2	1.0039
0.3	1.0128
0.4	1.0291
0.5	1.0537
0.6	1.0863
0.7	1.1258
0.8	1.1701
0.9	1.2168
1.0	1.2633

NUMERICAL INTEGRATION METHODS (Cont.)

Solution (Cont.)

Step 3

Apply a composite trapezoidal rule formula

$$\int_0^1 \left(\sqrt{\sin^3(x)+1}\right)dx$$

$$\cong \frac{0.1}{2}\left(1+1.2633+2\left(1.0005+1.0039+\ldots+1.2168\right)\right)$$

$$=\frac{0.1}{2}\left(2.2633+2\left(9.7012\right)\right)$$

$$=1.0833$$

Therefore, the numerical integration of Example 2 is 1.0833

NUMERICAL INTEGRATION METHODS (Cont.)

Solution (Cont.)

Step 4

Apply a composite trapezoidal rule formula

$$\int_0^1 \left(\sqrt{\sin^3(x)+1}\right)dx$$

$$\cong \frac{0.1}{2}\left(1+1.2633+2\left(1.0005+1.0039+\ldots+1.2168\right)\right)$$

$$=\frac{0.1}{2}\left(2.2633+2\left(9.7012\right)\right)$$

$$=1.0833$$

Therefore, the numerical integration of Example 2 is 1.0833

NUMERICAL INTEGRATION METHODS (Cont.)

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Simpson's Rule

- Higher order polynomial can be used to obtain more accurate estimate of the integral. If there is a mid point in between $f(a)$ and $f(b)$, then
- The **three points** can be connected with a **second order polynomial**
- The **four points** can be connected with a **third order polynomial**
- The numerical integration method that based on second and third order polynomials are called **Simpson's rule.**

NUMERICAL INTEGRATION METHODS (Cont.)

Simpson's Rule

Two Types of Methods

1

son's 1/3rd Rule

2

Simpson's 3/8 R

NUMERICAL INTEGRATION METHODS (Cont.)

Simpson's 1/3rd Rule

- Integrand is approximated by a second order polynomial
- Three points are connected with a parabola
- The integrand in equation (1) is substituted with a second order interpolation polynomial of

$$I \cong \int_a^b f_2(x)dx$$

where $f_2(x)$ is second order Lagrange interpolation polynomial.

NUMERICAL INTEGRATION METHODS (Cont.)

Single Application Simpson's 1/3rd Rule

Suppose $a = x_0$ and $b = x_n$, the integral of (1) can be written as

$$I = \int \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx \qquad (5)$$

By integrating (5) and with some algebraic manipulation yields

$$I = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2) \right]$$

Single application Simpson's 1/3rd rule

where $h = \frac{b-a}{2}$.

NUMERICAL INTEGRATION METHODS (Cont.)

Composite Simpson's 1/3rd Rule

- The accuracy of single application of Simpson's 1/3rd rule can be improved by dividing the interval into n number of strips of equal width
- The method is known as **Composite Simpson's 1/3rd rule.**
- The total integral can be expressed as

$$I = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \ldots + \int_{x_{n-2}}^{x_n} f(x)\,dx \qquad (6)$$

- The individual integral of (6) is substituted by single application of Simpson's 1/3rd rule such that

$$I = \frac{h}{3}\Big[\big(f(x_0)+4f(x_1)+f(x_2)\big)+\big(f(x_2)+4f(x_3)+f(x_4)\big)$$
$$+\ldots+\big(f(x_{n-2})+4f(x_{n-1})+f(x_n)\big)\Big] \qquad (7)$$

NUMERICAL INTEGRATION METHODS (Cont.)

Composite Simpson's 1/3rd Rule (Cont.)

By combining and rearranging terms of equation (7) gives

$$I = \frac{h}{3}\left[\left(f(x_0) + f(x_n) + 4\sum_{i=1,3,5,..}^{n-1} f(x_i) + 2\sum_{i=2,4,6,..}^{n-2} f(x_i)\right)\right]$$

Composite Simpson's 1/3rd Rule

Note: The number of segments, n should be an even number

NUMERICAL INTEGRATION METHODS (Cont.)

Example 3

Evaluate

$$\int_0^1 \left(\sqrt{\sin^3(x) + 1} \right) dx$$

by using Simpson's rule.

Solution

Single application of Simpson's 1/3rd rule is used, since no information of step size or number of strips is provided.

Step 1

$$b = 1, \; a = 0, \; n = 2$$

$$h = \frac{1-0}{2} = 0.5$$

Step size, h

NUMERICAL INTEGRATION METHODS (Cont.)

Solution (Cont.)

Step 2

For each value of x, find $f(x)$

x	$f(x)$
0	1.0
0.5	1.0537
1.0	1.2633

Step 3

$$I \cong \frac{0.5}{3}\left[1+4\left(1.0537\right)+1.2633\right]=1.0793$$

Therefore, the numerical integration is 1.0793

NUMERICAL INTEGRATION METHODS (Cont.)

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Example 3

Evaluate

$$\int_0^1 \left(\sqrt{\sin^3(x)+1}\right)dx$$

by using Simpson's rule with $n = 10$.

Solution

Single application of Simpson's 1/3rd rule is used, since no information of step size or number of strips is provided.

Step 1

$$b=1,\ a=0,\ n=10$$

$$h=\frac{1-0}{10}=0.1$$

Step size, h

NUMERICAL INTEGRATION METHODS (Cont.)

Solution (Cont.)

Step 2

For each value of x, find $f(x)$

n	x	$f(x_0), f(x_n)$	$f(x_{i_{even}})$	$f(x_{i_{odd}})$
0	0	1		
1	0.1			1.0005
2	0.2		1.0039	
3	0.3			1.0128
4	0.4		1.0291	
5	0.5			1.0537
6	0.6		1.0863	
7	0.7			1.1258
8	0.8		1.1701	
9	0.9			1.2168
10	1.0	1.2633		
Total		**2.2633**	**4.2894**	**5.4096**

NUMERICAL INTEGRATION METHODS (Cont.)

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Solution (Cont.)

Step 3

$$I \cong \frac{0.1}{3}\left[2.2633 + 2\left(4.2894\right) + 4\left(5.4096\right)\right] = 1.0827$$

Therefore, the numerical integration is 1.0827

NUMERICAL INTEGRATION METHODS (Cont.)

Simpson's 3/8 Rule

- Integrand is approximated by a third order Lagrange polynomial
- Require four points of data with number of intervals $n = 3$.
- The integrand in equation (1) is substituted with a third order Lagrange polynomial of

$$I \cong \int_a^b f_3(x)dx$$

where $f_3(x)$ is third order Lagrange interpolation polynomial.

NUMERICAL INTEGRATION METHODS (Cont.)

Composite Simpson’s 3/8 Rule Formula

Formula

$$I = \frac{3h}{8}\left[f\left(x_0\right) + 3f\left(x_1\right) + 3f\left(x_2\right) f\left(x_3\right) \right]$$

$$h = \frac{b-a}{3}$$

NUMERICAL INTEGRATION METHODS (Cont.)

Example 3

Evaluate

$$\int_0^1 \left(\sqrt{\sin^3(x) + 1} \right) dx$$

by using Simpson's 3/8 rule.

Solution

Three segments with four equally spaced points are required.

Step 1

$$b = 1,\ a = 0,\ n = 3$$

$$h = \frac{1-0}{3} = \frac{1}{3}$$

Step size, h

NUMERICAL INTEGRATION METHODS (Cont.)

Solution (Cont.)

Step 2

For each value of x, find $f(x)$

x	$f(x)$
0	1.0
$\frac{1}{3}$	1.0174
$\frac{2}{3}$	1.1120
1.0	1.2633

Step 3

$$I \cong \frac{3\left(\frac{1}{3}\right)}{8}\left[1+3(1.0174)+3(1.1120)+1.2633\right]=1.0814$$

Therefore, the numerical integration is 1.0814

NUMERICAL INTEGRATION METHODS (Cont.)

Simpson's 1/3rd Rule and 3/8 Rule in Tandem

- Simpson's 3/8 rule is implemented to approximate the integrand (1) with $n = 3$ and four number of points.
- Simpson's 1/3rd rule is limited for even number of segments.
- To permit the computation of integrand for odd number of segments, Simpson's 1/3rd and 3/8 rule can be applied in tandem.

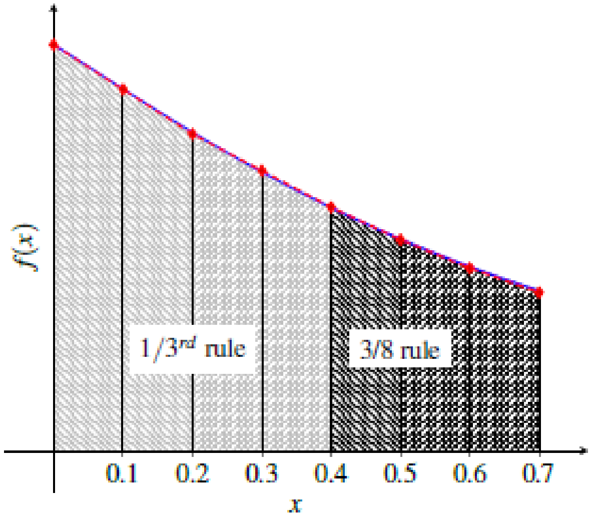

Figure 4: Graphical representation of the approximation of the integral with odd numbers of intervals by Simpson's 1/3rd and 3/8 rule.

NUMERICAL INTEGRATION METHODS (Cont.)

Example 3

Evaluate

$$\int_{0}^{1}\left(\sqrt{\sin^{3}(x)+1}\right)dx$$

by using Simpson's rule with $n = 5$.

Solution

Three segments with four equally spaced points are required.

Step 1

$$b = 1,\ a = 0,\ n = 5$$

$$h = \frac{1-0}{5} = 0.2$$

Step size, h

NUMERICAL INTEGRATION METHODS (Cont.)

Solution (Cont.)

Step 2

$$\int_0^1 \left(\sqrt{\sin^3(x)+1}\right)dx = \underbrace{\int_0^{0.4} \sqrt{\sin^3(x)+1}dx}_{Simpson's\,1/3^{rd}\ rule} + \underbrace{\int_{0.4}^{1} \sqrt{\sin^3(x)+1}dx}_{Simpson's\ 3/8\ rule}$$

Step 3

For each value of x, find $f(x)$

x	$f(x)$
0	1.0
0.2	1.0039
0.4	1.0291
0.6	1.0863
0.8	1.1701
1.0	1.2633

NUMERICAL INTEGRATION METHODS (Cont.)

Solution (Cont.)

Step 4

$$\int_0^{0.4} \sqrt{\sin^3(x)+1}\,dx = \frac{0.2}{3}\left(1+4\left(1.0039\right)+1.0291\right)$$

$$= 0.4030$$

$$\int_{0.4}^{1} \sqrt{\sin^3(x)+1}\,dx = \frac{3\left(0.2\right)}{8}\left(1.0291+3\left(1.0863\right)+3\left(1.1701\right)+1.2633\right)$$

$$= 0.6796$$

$$\int_0^{1} \left(\sqrt{\sin^3(x)+1}\right)dx = 0.4030+0.6796 = 1.0826$$

Therefore, the numerical integration is 1.0826

Conclusion

Integration Method	Single	Composite
Trapezoidal Rule	$I=\frac{h}{2}\left[f(a)+f(b)\right]$ where $h=b-a$	$I=\frac{h}{2}\left[f(x_0)+f(x_n)+2\sum_{i=1}^{n-1}f(x_i)\right]$ where $h=\frac{b-a}{n}$
Simpson's 1/3rd Rule	$I=\frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]$ where $h=\frac{b-a}{2}$	$I=\frac{h}{3}\left[f(x_0)+f(x_n)+4\underbrace{\left[f(x_1)+\ldots f(x_{n-1})\right]}_{\text{odd term}}+2\underbrace{\left[f(x_2)+\ldots f(x_{n-2})\right]}_{\text{even term}}\right]$ where $h=\frac{b-a}{n}$
Simpson's 3/8 Rule	$I=\frac{3h}{8}\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right]$ where $h=\frac{b-a}{3}$	

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