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Numerical Methods
Solving Linear Algebraic Equations: Iterative Methods

by

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Description

AIMS

This chapter is aimed to solve small numbers of linear algebraic equations by using iterative methods involving **Jacobi method** and **Gauss-Seidel method**

EXPECTED OUTCOMES

1. Students should be able to solve linear algebraic equations by using Jacobi and Gauss-Seidel methods

REFERENCES

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2. Chapra, C. S. & Canale, R. P. *Numerical Methods for Engineers*, Sixth Edition, McGraw–Hill, 2010.

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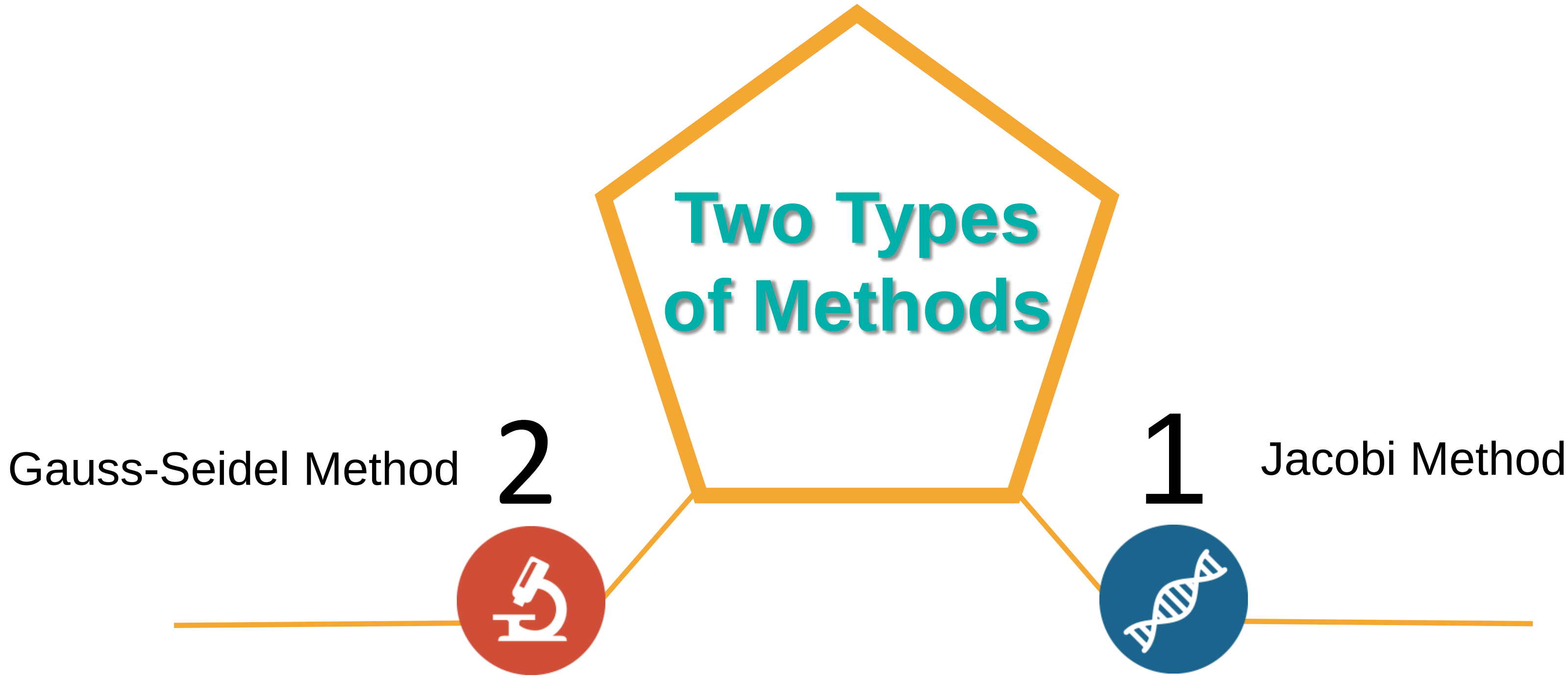
1. Introduction
2. Jacobi Method
3. Gauss-Seidel Method

INTRODUCTION

Solving Linear Algebraic Equations: Iterative Methods



JACOBI METHOD

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Jacobi Methods Procedures

STEP 1 Rearrange the equation to make the system **diagonally dominant**

STEP 2 Write the equation in an **explicit form**

STEP 3 The calculation process starts by assuming initial values for the unknown for the first iteration

JACOBI METHOD (Cont.)

Jacobi Methods Procedures (Cont.)

Step 1

Rearrange the equation to make the system diagonally dominant. For a system of n equations, $\mathbf{Ax} = \mathbf{b}$ a sufficient condition for convergence is that for every row of matrix, the absolute of the diagonal element is greater than or equal to the sum of the absolute values of the diagonal elements in that row

$$|a_{ii}| \geq \sum_{j=1,\, j\neq i}^{n} |a_{ij}|$$

JACOBI METHOD (Cont.)

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Jacobi Methods Procedures (Cont.)

Step 2

Write the equation in an explicit form

Each unknown is written in terms of the other unknowns.
For 3 x 3 matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1^{(k+1)} = \frac{b_1 - a_{12} x_2^{(k)} - a_{13} x_3^{(k)}}{a_{11}}$$

$$x_2^{(k+1)} = \frac{b_2 - a_{21} x_1^{(k)} - a_{23} x_3^{(k)}}{a_{22}}$$

$$x_3^{(k+1)} = \frac{b_3 - a_{31} x_1^{(k)} - a_{32} x_2^{(k)}}{a_{33}}$$

JACOBI METHOD (Cont.)

Jacobi Methods Procedures (Cont.)

Step 3

The calculation process starts by assuming initial values for the unknowns for the first iteration. Start the calculation by using the given initial condition.

JACOBI METHOD (Cont.)

Example 1

Use the Jacobi method to obtain the solution for.

$$8x_1 + x_2 + x_3 = 10$$
$$2x_1 + x_2 + 9x_3 = -2$$
$$x_1 - 7x_2 + 2x_3 = 4$$

Use 4 decimal places in your computation and let $x_i^{(0)} = (0,0,0)^T$. Compute up to two iterations and calculate the approximate percent relative error for each iteration.

Solution

Step 1 Rearrange the equation to make the system diagonally dominant.

$$\begin{bmatrix} 8 & 1 & 1 \\ 2 & 1 & 9 \\ 1 & -7 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 8 & 1 & 1 \\ 1 & -7 & 2 \\ 2 & 1 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ -2 \end{bmatrix}$$

JACOBI METHOD (Cont.)

Solution (Cont.)

Step 2

Write the equations in explicit form.

$$x_1^{(k+1)} = \frac{10 - x_2^{(k)} - x_3^{(k)}}{8}$$

$$x_2^{(k+1)} = \frac{4 - x_1^{(k)} - 2x_3^{(k)}}{-7}$$

$$x_3^{(k+1)} = \frac{-2 - 2x_1^{(k)} - x_2^{(k)}}{9}$$

JACOBI METHOD (Cont.)

Solution (Cont.)

Step 3

Start calculation with the given initial condition.

1st iteration, $k = 0, \quad x_i^{(0)} = (0,0,0)^T$

$$x_1^{(1)} = \frac{10 - x_2^{(0)} - x_3^{(0)}}{8} = \frac{10-0-0}{8} = 1.25$$

$$x_2^{(1)} = \frac{4 - x_1^{(0)} - 2x_3^{(0)}}{-7} = \frac{4-0-0}{-7} = -0.5714$$

$$x_3^{(1)} = \frac{-2 - 2x_1^{(0)} - x_2^{(0)}}{9} = \frac{-2-0-0}{9} = -0.2222$$

JACOBI METHOD (Cont.)

Solution (Cont.)

The approximate percent relative error for $x_1^{(1)}$, $x_2^{(1)}$, and $x_3^{(1)}$ are computed as follows:

$$\varepsilon_{a_{x_1}} = \left|\frac{1.25-0}{1.25}\right| \times 100\% = 100\%$$

$$\varepsilon_{a_{x_2}} = \left|\frac{-0.5714-0}{-0.5714}\right| \times 100\% = 100\%$$

$$\varepsilon_{a_{x_3}} = \left|\frac{-0.2222-0}{-0.2222}\right| \times 100\% = 100\%$$

JACOBI METHOD (Cont.)

Solution (Cont.)

2nd iteration, $k=1$, $x_i^{(1)} = (1.2500, -0.5714, -0.2222)^T$

$$x_1^{(2)} = \frac{10 - x_2^{(1)} - x_3^{(1)}}{8} = \frac{10 - (-0.5714) - (-0.2222)}{8} = 1.3492$$

$$x_2^{(2)} = \frac{4 - x_1^{(1)} - 2x_3^{(1)}}{-7} = \frac{4 - 1.25 + 0.4444}{-7} = -0.4563$$

$$x_3^{(2)} = \frac{-2 - 2x_1^{(1)} - x_2^{(1)}}{9} = \frac{-2 - 2.5 + 0.5714}{9} = -0.4365$$

JACOBI METHOD (Cont.)

Solution (Cont.)

The approximate percent relative error for $x_1^{(2)}$, $x_2^{(2)}$, and $x_3^{(2)}$ are computed as follows:

$$\varepsilon_{a_{x_1}} = \left|\frac{1.3492 - 1.25}{1.3492}\right| \times 100\% = 7.35\%$$

$$\varepsilon_{a_{x_2}} = \left|\frac{-0.4563 + 0.5714}{-0.4563}\right| \times 100\% = 25.22\%$$

$$\varepsilon_{a_{x_3}} = \left|\frac{-0.4365 + 0.2222}{-0.4365}\right| \times 100\% = 49.10\%$$

Therefore after two iterations, $x_1 = 1.3492$, $x_2 = -0.4563$, $x_3 = -0.4365$.

GAUSS-SEIDEL METHOD (Cont.)

Gauss Seidel Methods Procedures

STEP 1

Rearrange the equation to make the system **diagonally dominant**

STEP 2

Write the equation in an **explicit form**

$$x_1^{(k+1)} = \frac{b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}}{a_{11}}$$

$$x_2^{(k+1)} = \frac{b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}}{a_{22}}$$

$$x_3^{(k+1)} = \frac{b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}}{a_{33}}$$

STEP 3

The calculation process starts by assuming initial values for the unknown for the first iteration

GAUSS-SEIDEL METHOD (Cont.)

Example 2

Use the Gauss-Seidel method to obtain the solution for.

$$8x_1 + x_2 + x_3 = 10$$

$$2x_1 + x_2 + 9x_3 = -2$$

$$x_1 - 7x_2 + 2x_3 = 4$$

Use 4 decimal places in your computation and let $x_i^{(0)} = (0,0,0)^T$. Compute up to two iterations and calculate the approximate percent relative error for each iteration and compare the solution with the results obtain in Example 1.

Solution

Step 1

Rearrange the equation to make the system diagonally dominant.

$$\begin{bmatrix} 8 & 1 & 1 \\ 2 & 1 & 9 \\ 1 & -7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 8 & 1 & 1 \\ 1 & -7 & 2 \\ 2 & 1 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ -2 \end{bmatrix}$$

GAUSS-SEIDEL METHOD (Cont.)

Solution (Cont.)

Step 2

Write the equations in explicit form.

$$x_1^{(k+1)} = \frac{10 - x_2^{(k)} - x_3^{(k)}}{8}$$

$$x_2^{(k+1)} = \frac{4 - x_1^{(k+1)} - 2x_3^{(k)}}{-7}$$

$$x_3^{(k+1)} = \frac{-2 - 2x_1^{(k+1)} - x_2^{(k+1)}}{9}$$

GAUSS-SEIDEL METHOD (Cont.)

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Solution (Cont.)

Step 3

Start calculation with the given initial condition.

1st iteration, $k = 0, \quad x_i^{(0)} = (0,0,0)^T$

$$x_1^{(1)} = \frac{10 - x_2^{(0)} - x_3^{(0)}}{8} = \frac{10-0-0}{8} = 1.25$$

$$x_2^{(1)} = \frac{4 - x_1^{(1)} - 2x_3^{(0)}}{-7} = \frac{4-1.25-0}{-7} = -0.3929$$

$$x_3^{(1)} = \frac{-2 - 2x_1^{(1)} - x_2^{(1)}}{9} = \frac{-2-2(1.25)+0.3929}{9} = -0.4563$$

GAUSS-SEIDEL METHOD (Cont.)

Solution (Cont.)

The approximate percent relative error for $x_1^{(1)}$, $x_2^{(1)}$, and $x_3^{(1)}$ are computed as follows:

$$\varepsilon_{a_{x_1}} = \left|\frac{1.25-0}{1.25}\right| \times 100\% = 100\%$$

$$\varepsilon_{a_{x_2}} = \left|\frac{-0.3929-0}{-0.3929}\right| \times 100\% = 100\%$$

$$\varepsilon_{a_{x_3}} = \left|\frac{-0.4563-0}{-0.4563}\right| \times 100\% = 100\%$$

GAUSS-SEIDEL METHOD (Cont.)

Solution (Cont.)

2nd iteration, $k = 1$, $x_i^{(1)} = (1.2500, -0.3929, -0.4563)^T$

$$x_1^{(2)} = \frac{10 - x_2^{(1)} - x_3^{(1)}}{8} = \frac{10 - (-0.3929) - (-0.4563)}{8} = 1.3562$$

$$x_2^{(2)} = \frac{4 - x_1^{(2)} - 2x_3^{(1)}}{-7} = \frac{4 - 1.3562 - 2(-0.4563)}{-7} = -0.5081$$

$$x_3^{(2)} = \frac{-2 - 2x_1^{(2)} - x_2^{(2)}}{9} = \frac{-2 - 2(1.3562) + 0.5081}{9} = -0.4671$$

GAUSS-SEIDEL METHOD (Cont.)

Solution (Cont.)

The approximate percent relative error for $x_1^{(2)}, x_2^{(2)}$, and $x_3^{(2)}$ are computed as follows:

$$\varepsilon_{a_{x_1}} = \left|\frac{1.3562 - 1.25}{1.3562}\right| \times 100\% = 7.83\%$$

$$\varepsilon_{a_{x_2}} = \left|\frac{-0.5081 + 0.3929}{-0.5081}\right| \times 100\% = 22.67\%$$

$$\varepsilon_{a_{x_3}} = \left|\frac{-0.4671 + 0.4563}{-0.4671}\right| \times 100\% = 2.31\%$$

Therefore after two iterations, $x_1 = 1.3562$, $x_2 = -0.5081$, $x_3 = -0.4671$.

GAUSS-SEIDEL METHOD (Cont.)

Solution (Cont.)

Comparison between Gauss-Seidel and Jacobi methods of this example is presented below:

Variable	Jacobi Methods	$\varepsilon_{a_{Jacobi}}(\%)$	Gauss-Seidel Method	$\varepsilon_{a_{Gauss-Sedel}}(\%)$	Exact Solution
x_1	1.3492	7.35	1.3562	7.83	1.3725
x_2	-0.4563	25.22	-0.5081	22.67	-0.5098
x_3	-0.4365	49.10	-0.4671	2.31	-0.4706

The solution that is obtained via Gauss-Seidel method converge faster compare to Jacobi Method.

Conclusion

Method	Formula
Jacobi	$x_1^{(k+1)} = \dfrac{b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}}{a_{11}}$ $x_2^{(k+1)} = \dfrac{b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)}}{a_{22}}$ $x_3^{(k+1)} = \dfrac{b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)}}{a_{33}}$
Gauss-Seidel	$x_1^{(k+1)} = \dfrac{b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}}{a_{11}}$ $x_2^{(k+1)} = \dfrac{b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}}{a_{22}}$ $x_3^{(k+1)} = \dfrac{b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}}{a_{33}}$

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