Numerical Methods
Solving Linear Algebraic Equations: Direct Methods

by

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Description

AIMS

This chapter is aimed to solve small numbers of linear algebraic equations by using direct methods involving **Naïve Gauss elimination method, Gauss elimination as partial pivoting** and **LU factorization methods**

EXPECTED OUTCOMES

1. Students should be able transform the system of linear equations into matrix form.
2. Students should be able to solve linear algebraic equations by using Naïve Gauss elimination method, Gauss elimination as partial pivoting and LU factorization.

REFERENCES

1. Norhayati Rosli, Nadirah Mohd Nasir, Mohd Zuki Salleh, Rozieana Khairuddin, Nurfatihah Mohamad Hanafi, Noraziah Adzhar. *Numerical Methods,* Second Edition, UMP, 2017 (Internal use)
2. Chapra, C. S. & Canale, R. P. *Numerical Methods for Engineers*, Sixth Edition, McGraw–Hill, 2010.

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2. Naïve Gauss Elimination Method
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INTRODUCTION

Solving Linear Algebraic Equations: Direct Methods

Three Types of Methods

1. Naïve Gauss Elimination Method
2. Gauss Elimination as Partial Pivoting
3. LU Factorization

NAÏVE GAUSS ELIMINATION

Gauss elimination is a sequential process of eliminate unknowns from a system of equations by using **forward elimination** and solving for the unknown by using **backward substitution**.

Two Major Steps

1 Forward Elimination

2 Backward Substitution

NAÏVE GAUSS ELIMINATION (Cont.)

Naïve Gauss Elimination Procedures

STEP 1 A system of linear equations is written in **augmented** matrix

STEP 2 Identify **pivot** element. The pivot element is used as multiplier to derive row operation formula

STEP 3 Perform a **forward elimination** to transform the matrix to an upper triangular matrix

STEP 4 Solve the system by using **backward substitution**

NAÏVE GAUSS ELIMINATION (Cont.)

Forward Elimination

In forward elimination step the augmented matrix of $[\mathbf{A}|\mathbf{b}]$ is reduced to an upper triangular matrix.

Forward Elimination Procedure

Step 1

Rewrite a system of linear equations in a augmented matrix, $[\mathbf{A}|\mathbf{b}]$. Let consider n–numbers of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

Augmented Matrix

Pivot element

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & b_n \end{array}\right]$$

NAÏVE GAUSS ELIMINATION (Cont.)

Forward Elimination Procedure (Cont.)

Step 2

Transform the augmented matrix to an upper triangular matrix. This transformation is executing by **forward elimination of unknowns**. The initial step in forward elimination of unknowns is to eliminate, the first unknown x_1 from the second through n^{th} equations by using the multiplier and its row operation formula as stated in **Table 1**. The new augmented matrix becomes

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & a'_{n2} & \cdots & a'_{nn} & b'_n \end{array}\right]$$

where the prime indicates that the elements have been changed from its original values.

NAÏVE GAUSS ELIMINATION (Cont.)

Forward Elimination Procedure (Cont.)

Step 3

The elimination procedure is repeated for the remaining equations to eliminate, the second unknown x_2 from the second through n^{th} equations. The new augmented matrix becomes

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} & b'_2 \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} & b''_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a''_{nn} & b'_n \end{array}\right]$$

where the double prime indicates that the elements have been modified twice.

NAÏVE GAUSS ELIMINATION (Cont.)

Forward Elimination Procedure (Cont.)

Step 4

The final manipulation in the sequence is to use the $(n-1)^{th}$ equation to eliminate the term x_{n-1} from the n^{th} equation. The system is then transformed to an upper triangular matrix, $\mathbf{U}$.

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} & b'_2 \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} & b''_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{(n-1)} & b_n^{(n-1)} \end{array}\right]$$

Upper triangular matrix

NAÏVE GAUSS ELIMINATION (Cont.)

Table 1: Row operation for each step in forward elimination procedure of $n \times n$ system.

Step	Row	Multiplier	Row Operation
1	$i = 2,3, \ldots, n$	$m_{i1} = -\dfrac{a_{i1}}{a_{11}}$	$R_i + m_{i1}R_1$
2	$i = 3,4, \ldots, n$	$m_{i2} = -\dfrac{a_{i2}}{a_{22}}$	$R_i + m_{i2}R_2$
3	$i = 4,5, \ldots, n$	$m_{i3} = -\dfrac{a_{i3}}{a_{33}}$	$R_i + m_{i3}R_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4	$i = n$	$m_{i(n-1)} = -\dfrac{a_{i(n-1)}}{a_{(n-1)(n-1)}}$	$R_i + m_{i(n-1)}R_{n-1}$

NAÏVE GAUSS ELIMINATION (Cont.)

Backward Substitution

Starting with the last row, solve for the unknown. The results obtained is back substituted into $(n-1)$ equation to solve for x_{n-1} by using the following formula.

The n^{th} equation can be solved by using:

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

The result is substituted into $(n-1)$ equation to solve the remaining x by using:

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}}, \quad i = n-1, n-2, \ldots, 1$$

NAÏVE GAUSS ELIMINATION (Cont.)

Example 1

Solve the following system of linear equations using Naïve Gauss elimination method.

$$4x_1 + 3x_2 + 2x_3 = 13$$
$$2x_1 - x_2 + x_3 = -1$$
$$x_1 + x_2 + 3x_3 = 14$$

Solution

Forward elimination of unknowns:

$$\left[\begin{array}{ccc|c} 4 & 3 & 2 & 13 \\ 2 & -1 & 1 & -1 \\ 1 & 1 & 3 & 14 \end{array}\right] \xrightarrow[R_3+\left(\frac{-1}{4}\right)R_1]{R_2+\left(\frac{-2}{4}\right)R_1} \left[\begin{array}{ccc|c} 4 & 3 & 2 & 13 \\ 0 & -\frac{5}{2} & 0 & -\frac{15}{2} \\ 0 & \frac{1}{4} & \frac{5}{2} & \frac{43}{4} \end{array}\right] \xrightarrow[R_3+\left(\frac{-1}{4}\right)R_1]{R_2+\left(\frac{-2}{4}\right)R_1} \left[\begin{array}{ccc|c} 4 & 3 & 2 & 13 \\ 0 & -\frac{5}{2} & 0 & -\frac{15}{2} \\ 0 & 0 & \frac{5}{2} & 10 \end{array}\right]$$

NAÏVE GAUSS ELIMINATION (Cont.)

Solution (Cont.)

Backward substitution:

$$\frac{5}{2}x_3 = 10 \Rightarrow x_3 = 4$$

$$-\frac{5}{2}x_2 = -\frac{15}{2} \Rightarrow x_2 = 3$$

$$4x_1 + 3(3) + 2(4) = 13 \Rightarrow x_1 = -1$$

Therefore, $x_1 = -1, x_2 = 3,\ x_3 = 4.$

GAUSS ELIMINATION WITH PARTIAL PIVOTING

Two **pitfalls** involve in Naïve Gauss elimination method:

a) Division by zero – may occur in forward elimination and backward substitution phases.
b) Large round off errors

GAUSS ELIMINATION WITH PARTIAL PIVOTING (Cont.)

- The simplest remedy to reduce the round–off error is to increase the number of significant figures in the computation. However, it would contribute to the computational effort and large memory are required for the storage.
- The technique of pivoting can be used to avoid both division by zero and minimizes round–off error.
- Pivoting involve the rearrangement of the equations so that the pivot element has the largest magnitude compare to others element in that similar column.
- This technique can be incorporated into Naive Gauss elimination method and it is frequently known as **Gauss elimination with partial pivoting**. It involves the same step as Naïve gauss elimination method (forward elimination and backward substitution) with an additional task that we need to ensure the pivot element must have the largest magnitude before each of the $(n-1)$ steps of forward elimination is performed.

GAUSS ELIMINATION WITH PARTIAL PIVOTING (Cont.)

Example 2 [Chapra & Canale, 2010]

Solve the following equations using Gauss elimination with partial pivoting.

$$0.3x_1 + 0.52x_2 + x_3 = -0.01$$
$$0.5x_1 + x_2 + 1.9x_3 = 0.67$$
$$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$$

Solution

Rewrite in augmented matrix form

$$\left[\begin{array}{ccc|c} 0.3 & 0.52 & 1 & -0.01 \\ 0.5 & 1 & 1.9 & 0.67 \\ 0.1 & 0.3 & 0.5 & -0.44 \end{array}\right]$$

GAUSS ELIMINATION WITH PARTIAL PIVOTING (Cont.)

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Solution (Cont.)

Identify the largest absolute value in the first column and switch between the two rows (if any).

Pivot element

$$\left[\begin{array}{ccc|c} 0.3 & 0.52 & 1 & -0.01 \\ 0.5 & 1 & 1.9 & 0.67 \\ 0.1 & 0.3 & 0.5 & -0.44 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 0.5 & 1 & 1.9 & 0.67 \\ 0.3 & 0.52 & 1 & -0.01 \\ 0.1 & 0.3 & 0.5 & -0.44 \end{array}\right]$$

Forward elimination of unknowns:

$$\left[\begin{array}{ccc|c} 0.5 & 1 & 1.9 & 0.67 \\ 0.3 & 0.52 & 1 & -0.01 \\ 0.1 & 0.3 & 0.5 & -0.44 \end{array}\right] \xrightarrow[R_3' = R_3 + \left(-\frac{0.1}{0.5}\right) R_1]{R_2' = R_2 + \left(-\frac{0.3}{0.5}\right) R_1} \left[\begin{array}{ccc|c} 0.5 & 1 & 1.9 & 0.67 \\ 0 & -0.0800 & -0.1400 & -0.4120 \\ 0 & 0.1000 & 0.1200 & -0.5740 \end{array}\right]$$

GAUSS ELIMINATION WITH PARTIAL PIVOTING (Cont.)

Solution (Cont.)

Identify the largest absolute value in second column and switch between the two rows (if any).

$$\left[\begin{array}{ccc|c} 0.5 & 1 & 1.9 & 0.67 \\ 0 & -0.0800 & -0.1400 & -0.4120 \\ 0 & 0.1000 & 0.1200 & -0.5740 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 0.5 & 1 & 1.9 & 0.67 \\ 0 & 0.1000 & 0.1200 & -0.5740 \\ 0 & -0.0800 & -0.1400 & -0.4120 \end{array}\right]$$

Forward elimination of unknowns:

$$\left[\begin{array}{ccc|c} 0.5 & 1 & 1.9 & 0.67 \\ 0 & 0.1000 & 0.1200 & -0.5740 \\ 0 & -0.0800 & -0.1400 & -0.4120 \end{array}\right] \xrightarrow{R_3' = R_3 + \left(-\frac{(-0.0800)}{0.1000}\right) R_2} \left[\begin{array}{ccc|c} 0.5 & 1 & 1.9 & 0.67 \\ 0 & 0.1000 & 0.1200 & -0.5740 \\ 0 & 0 & -0.0440 & -0.8712 \end{array}\right]$$

GAUSS ELIMINATION WITH PARTIAL PIVOTING (Cont.)

Solution (Cont.)

Backward substitution:

$$-0.0440\,x_3 = -0.8712 \;\Rightarrow\; x_3 = 19.8000$$

$$0.1000\,x_2 + 0.1200\,x_3 = -0.5740$$

$$x_2 = \frac{-0.5740 - 0.1200\,(19.8000)}{0.1000} \Rightarrow x_2 = -29.5000$$

$$0.5x_1 + x_2 + 1.9x_3 = 0.67$$

$$x_1 = \frac{0.67 - (-29.5000) - 1.9(19.8073)}{0.5} \Rightarrow x_1 = -14.9000$$

Therefore, $x_1 = -14.9000, x_2 = 29.5000,\ x_3 = 19.8000.$

LU FACTORIZATION

- $\mathbf{LU}$ factorization is designed so that the elimination step involves only operations on the matrix $\mathbf{A}$.
- It is well suited with the situations where many right hand side of vectors $\mathbf{b}$ need to be evaluated for a single matrix $\mathbf{A}$.
- The matrix $\mathbf{A}$ is decomposed (factored) into a product of two matrices $\mathbf{L}$ and $\mathbf{U}$ such that

$$\mathbf{A} = \mathbf{LU}$$

LU FACTORIZATION (Cont.)

- For 3 × 3 matrices we may have

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} & a_{31} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- By this decomposition, the system of $\mathbf{Ax} = \mathbf{b}$ to be solved has the form of

$$\mathbf{LUx} = \mathbf{b} \qquad (1)$$

- To solve equation (1), let consider

$$\mathbf{Ux} = \mathbf{d} \qquad (2)$$

- Substituted equation (2) into equation (1) gives

$$\mathbf{Ld} = \mathbf{b} \qquad (3)$$

LU FACTORIZATION (Cont.)

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LU Factorization Procedures

Step 1

LU decomposition– the matrix **A** is decomposed (factored) into a product of two matrices **L** (lower triangular matrix) and **U** (upper triangular matrix).

Step 2

Substitution– this steps consists of two steps which are forward and backward substitution. In forward substitution, equation (3) is used to generate an intermediate vector **d**. Then, the vector **d** is substituted in equation (2) to obtain the solution of **x** in backward substitution step.

LU FACTORIZATION (Cont.)

LU Factorization Methods

Three Types of Methods

1. Gauss Elimination Method as LU
2. Crout's Method
3. Cholesky Method

GAUSS ELIMATION METHOD AS LU FACTORIZATION

- Also known as a Doolittle decomposition (the elements of lower triangular matrix are 1’s on the main diagonal).

- Gauss elimination as $\mathbf{LU}$ decomposition and Naive Gauss elimination are equally efficient.

- Gauss elimination as $\mathbf{LU}$ decomposition is more time–consuming since the elimination step is done only on the matrix $\mathbf{A}$.

- Forward elimination step in Naive Gauss elimination method can be used to decompose $\mathbf{A}$ into $\mathbf{L}$ and $\mathbf{U}$.

GAUSS ELIMATION METHOD AS LU FACTORIZATION

Gauss Elimination as LU Factorization Procedures

From forward elimination step, we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{31} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow[R_3+R_1\left(-\frac{a_{31}}{a_{11}}\right)]{R_2+R_1\left(-\frac{a_{21}}{a_{11}}\right)} \begin{bmatrix} a_{11} & a_{12} & a_{31} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix} \xrightarrow{R_3+R_2\left(-\frac{a'_{32}}{a'_{22}}\right)} \begin{bmatrix} a_{11} & a_{12} & a_{31} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} = \mathbf{U}$$

GAUSS ELIMATION METHOD AS LU FACTORIZATION (Cont.)

Gauss Elimination as LU Factorization Procedures

Matrix **L** can be obtained from the multiplier of each row operation. Therefore, we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{31} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & -m_{32} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{31} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

where $m_{21} = \frac{a_{21}}{a_{11}}, m_{31} = \frac{a_{31}}{a_{11}}$ and $m_{32} = \frac{a'_{32}}{a'_{22}}$

GAUSS ELIMATION METHOD AS LU FACTORIZATION (Cont.)

Example 3

Solve the following equations using Gauss elimination as LU Factorization.

$$4x+3y+2z=13$$
$$2x-y+z=-1$$
$$x+y+3z=14$$

Solution

LU decomposition

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & -1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow[R_3+R_1\left(-\frac{1}{4}\right)]{R_2+R_1\left(-\frac{2}{4}\right)} \begin{bmatrix} 4 & 3 & 2 \\ 0 & -\frac{5}{2} & 0 \\ 0 & \frac{1}{4} & \frac{5}{2} \end{bmatrix} \xrightarrow{R_3+R_2\left(-\frac{1/4}{(-5/2)}\right)} \begin{bmatrix} 4 & 3 & 2 \\ 0 & -\frac{5}{2} & 0 \\ 0 & 0 & \frac{5}{2} \end{bmatrix} = U$$

GAUSS ELIMATION METHOD AS LU FACTORIZATION (Cont.)

Solution (Cont.)

Then write

$$\mathbf{U} = \begin{bmatrix} 4 & 3 & 2 \\ 0 & -\frac{5}{2} & 0 \\ 0 & 0 & \frac{5}{2} \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & -m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & -\frac{1}{10} & 1 \end{bmatrix}$$

GAUSS ELIMATION METHOD AS LU FACTORIZATION (Cont.)

Solution (Cont.)

Substitution to solve for $\mathbf{x}$

Set up $\mathbf{Ld} = \mathbf{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & -1/10 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 13 \\ -1 \\ 14 \end{bmatrix}$$

Solve for $\mathbf{d}$ by using forward substitution

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 13 \\ -15/2 \\ 10 \end{bmatrix}$$

GAUSS ELIMATION METHOD AS LU FACTORIZATION (Cont.)

Solution (Cont.)

Set up $\mathbf{Ux} = \mathbf{d}$

$$\begin{bmatrix} 4 & 3 & 2 \\ 0 & -5/2 & 0 \\ 0 & 0 & 5/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ -15/2 \\ 10 \end{bmatrix}$$

Solve for $\mathbf{x}$ by using backward substitution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$$

CROUT'S METHOD

- Executing the matrix multiplication on the right-hand side of the equation gives:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} l_{11} & (l_{11}u_{12}) & (l_{11}u_{13}) & (l_{11}u_{14}) \\ l_{21} & (l_{21}u_{12}+l_{22}) & (l_{21}u_{13}+l_{22}u_{23}) & (l_{21}u_{14}+l_{22}u_{24}) \\ l_{31} & (l_{31}u_{12}+l_{32}) & (l_{31}u_{13}+l_{32}u_{23}+l_{33}) & (l_{31}u_{14}+l_{32}u_{24}+l_{33}u_{34}) \\ l_{41} & (l_{41}u_{12}+l_{42}) & (l_{41}u_{13}+l_{42}u_{23}+l_{43}) & (l_{41}u_{14}+l_{42}u_{24}+l_{43}u_{34}+l_{44}) \end{bmatrix} --(*)$$

CROUT'S METHOD (Cont.)

- The elements of the matrices **L** and **U** can be determined by solving equation (*) by equating the corresponding elements of the matrix **A**.

First Row

$i = 1$

$$l_{11} = a_{11}$$

$$u_{13} = \frac{a_{13}}{l_{11}} \qquad u_{12} = \frac{a_{12}}{l_{11}} \qquad u_{14} = \frac{a_{14}}{l_{11}}$$

Second Row

$i = 2$

$$l_{21} = a_{21} \qquad l_{22} = a_{22} - l_{21}u_{12}$$

$$u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}} \qquad u_{24} = \frac{a_{24} - l_{21}u_{14}}{l_{22}}$$

Third Row

$i = 3$

$$l_{31} = a_{31} \qquad l_{32} = a_{32} - l_{31}u_{12} \qquad l_{33} = a_{33} - l_{31}u_{13}$$

$$u_{34} = \frac{a_{34} - l_{31}u_{14} - l_{32}u_{24}}{l_{33}}$$

Forth Row

$i = 4$

$$l_{41} = a_{41} \qquad l_{42} = a_{42} - l_{41}u_{12}$$

$$l_{43} = a_{43} - l_{41}u_{13} - l_{42}u_{23} \qquad l_{44} = a_{44} - l_{41}u_{14} - l_{42}u_{24} - l_{43}u_{34}$$

CROUT'S METHOD (Cont.)

Example 4

Solve the following equations using Crout's method.

$$12x - 2y + z = 16$$

$$15x + 12y - z = 1$$

$$13x + 4y + z = 11$$

Solution

LU decomposition

$$\begin{bmatrix} 12 & -2 & 1 \\ 15 & 12 & -1 \\ 13 & 4 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{23} + l_{33} \end{bmatrix}$$

CROUT'S METHOD (Cont.)

Solution (Cont.)

Equating the elements of the matrix A with the right–hand side of the multiplication of **L** and **U**, yield;

For $i = 1$ (first row)

$$l_{11} = 12$$

$$u_{12} = -\frac{1}{6}$$

$$u_{13} = \frac{1}{12}$$

For $i = 2$ (second row)

$$l_{21} = 15$$

$$l_{22} = \frac{29}{2}$$

$$u_{23} = -\frac{9}{58}$$

For $i = 3$ (third row)

$$l_{31} = 13$$

$$l_{32} = \frac{37}{6}$$

$$l_{33} = \frac{76}{87}$$

CROUT'S METHOD (Cont.)

Solution (Cont.)

$$\mathbf{U} = \begin{bmatrix} 1 & -\frac{1}{6} & \frac{1}{12} \\ 0 & 1 & -\frac{9}{58} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{L} = \begin{bmatrix} 12 & 0 & 0 \\ 15 & \frac{29}{2} & 0 \\ 13 & \frac{37}{6} & \frac{76}{87} \end{bmatrix}$$

CROUT'S METHOD (Cont.)

Solution (Cont.)

Substitution to solve for $\mathbf{x}$

Set up $\mathbf{Ld} = \mathbf{b}$

$$\begin{bmatrix} 12 & 0 & 0 \\ 15 & \frac{29}{2} & 0 \\ 13 & \frac{37}{6} & \frac{76}{87} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 1 \\ 11 \end{bmatrix}$$

Solve for $\mathbf{d}$ by using forward substitution

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ -\frac{38}{29} \\ 2 \end{bmatrix}$$

CROUT’S METHOD (Cont.)

Solution (Cont.)

Set up $\mathbf{Ux} = \mathbf{d}$

$$\begin{bmatrix} 1 & -\frac{1}{6} & \frac{1}{12} \\ 0 & 1 & -\frac{9}{58} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ -\frac{38}{29} \\ 2 \end{bmatrix}$$

Solve for $\mathbf{x}$ by using backward substitution, we obtain

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

CHOLESKY METHOD

Cholesky Method: Introduction

- Cholesky decomposition is a technique that is designed for a system where the matrix $\mathbf{A}$ is symmetric and positive definite.
- The symmetric matrix refers to the matrix with the element of $a_{ij} = a_{ji}$ for all $i \neq j$. In other words, $\mathbf{A} = \mathbf{A}^T$.
- Cholesky decomposition method offer computational advantages because only half of the storage and computation time are required.
- In Cholesky method, a symmetric matrix $\mathbf{A}$ is decomposed as

$$\mathbf{A} = \mathbf{U}^T\mathbf{U}$$

CHOLESKY METHOD (Cont.)

Cholesky Method: Formula

Cholesky Method Formula

$$u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2}$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj}}{u_{ii}} \quad \textbf{for } j = i+1, \ldots, n$$

CHOLESKY METHOD (Cont.)

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Cholesky Method: Procedures

Step 1

Decompose $\mathbf{A}$ such that $\mathbf{A} = \mathbf{U}^T\mathbf{U}$. Hence, we may have $\mathbf{U}^T\mathbf{U}\mathbf{x} = \mathbf{b}$.

Step 2

Set up and solve $\mathbf{U}^T\mathbf{d} = \mathbf{b}$, where $\mathbf{d}$ can be obtained by using forward substitution.

Step 3

Set up and solve $\mathbf{U}\mathbf{x} = \mathbf{d}$, where $\mathbf{x}$ can be obtained by using backward substitution

CHOLESKY METHOD (Cont.)

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Example 5

Solve the following equations using Cholesky Factorization.

$$2x - y = 1$$
$$-x = -2y + z$$
$$z = y$$

Solution

In Matrix Form

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

CHOLESKY METHOD (Cont.)

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Solution (Cont.)

LU decomposition

For the first row $(i=1)$:

$$u_{11}=\sqrt{a_{11}-\sum_{k=1}^{1-1}u_{k1}^{2}}=\sqrt{2}=1.4142$$

$$u_{12}=\frac{a_{12}-\sum_{k=1}^{0}u_{k1}u_{k2}}{u_{11}}=\frac{a_{12}}{u_{11}}=-0.7071$$

$$u_{13}=\frac{a_{13}-\sum_{k=1}^{0}u_{k1}u_{k3}}{u_{11}}=\frac{a_{13}}{u_{11}}=0$$

CHOLESKY METHOD (Cont.)

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Solution (Cont.)

LU decomposition

For the second row $(i = 2)$:

$$u_{22} = \sqrt{a_{22} - \sum_{k=1}^{2-1} u_{12}^2} = 1.2247$$

$$u_{23} = \frac{a_{23} - \sum_{k=1}^{1} u_{k2} u_{k3}}{u_{22}} = \frac{a_{23} - u_{12} u_{13}}{u_{22}} = -0.8165$$

For the third row $(i = 3)$:

$$u_{33} = \sqrt{a_{33} - \sum_{k=1}^{2} u_{k3}^2} = \sqrt{a_{33} - \left(u_{13}^2 + u_{23}^2\right)} = 0.5773$$

CHOLESKY METHOD (Cont.)

Solution (Cont.)

Therefore

$$\mathbf{U} = \begin{bmatrix} 1.4142 & -0.7071 & 0 \\ 0 & 1.2247 & -0.8165 \\ 0 & 0 & 0.5773 \end{bmatrix}$$

$$\mathbf{U}^T = \begin{bmatrix} 1.4142 & 0 & 0 \\ -0.7071 & 1.2247 & 0 \\ 0 & -0.8165 & 0.5773 \end{bmatrix}$$

CHOLESKY METHOD (Cont.)

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Solution (Cont.)

Substitution to solve for $\mathbf{x}$

Solve for $\mathbf{d}$ by using forward substitution

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 0.4083 \\ 0.5775 \end{bmatrix}$$

Solve for $\mathbf{x}$ by using backward substitution

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1.0002 \\ 1.0003 \\ 1.0003 \end{bmatrix}$$

Conclusion

- Naïve Gauss elimination method suffers from two pitfall which are may involve **division by zero** and **large round-off error.**
- Later can be overcome by performing **Gauss elimination method with partial pivoting**.
- $\mathbf{LU}$ factorization is designed so that the elimination step involves only operations on the matrix $\mathbf{A}$.
- It is well suited with the situations where many right hand side of vectors $\mathbf{b}$ need to be evaluated for a single matrix $\mathbf{A}$.

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