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Numerical Methods
Solving Small Numbers of Linear Algebraic Equations

By

Norhayati Rosli & Mohd Zuki Bin Salleh
Faculty of Industrial Sciences & Technology
norhayati@ump.edu.my, zuki@ump.edu.my

Description

AIMS

This chapter is aimed to solve small numbers of linear algebraic equations by using two types of methods involving **graphical method** and **Cramer's rule.**

EXPECTED OUTCOMES

1. Students should be able to transform the system of linear equations into matrix form.
2. Students should be able to solve small numbers of equations by using graphical method and Cramer's rule.

REFERENCES

1. Norhayati Rosli, Nadirah Mohd Nasir, Mohd Zuki Salleh, Rozieana Khairuddin, Nurfatihah Mohamad Hanafi, Noraziah Adzhar. *Numerical Methods,* Second Edition, UMP, 2017 (Internal use)
2. Chapra, C. S. & Canale, R. P. *Numerical Methods for Engineers*, Sixth Edition, McGraw–Hill, 2010.

Content

1. Introduction to System of Linear Equations
2. Graphical Method
3. Cramer’s Rule

INTRODUCTION TO SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

- System is defined as a group of entity or objects, real or abstract, comprising a whole with each and every component/element interacting or related to one another. For example, solar system, blood system and computer system.

- Mathematically, system of linear equations is a **collection of linear equations involving the same set of variables**

INTRODUCTION (Cont.)

System of linear equations is written as

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{m}x_n = b_n$$

INTRODUCTION (Cont.)

It can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

INTRODUCTION (Cont.)

In Augmented Matrix →

$$\left[\mathbf{A} \middle| \mathbf{b}\right] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right]$$

INTRODUCTION (Cont.)

Solving Small Number of Equations

Two Types of Methods

1. Graphical Method
2. Cramer's Rule

GRAPHICAL METHOD

- The graphical method is suitable to be applied to a linear algebraic system of two equations.
- These equations are plotting on **Cartesian Coordinate**.
- Each equation represents a straight line.
- The intersection point provides the solution to the corresponding unknown.
- For two equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

- Solve both equations for x_2 yields

$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} \quad \Rightarrow \quad x_2 = (\text{slope})\,x_1 + \text{intercept}$$

$$x_2 = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$

GRAPHICAL METHOD (Cont.)

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Example 1

Use the graphical method to solve the following equations

$$x_1 + 2x_2 = 7$$
$$-2x_1 + 3x_2 = 7$$

Solution

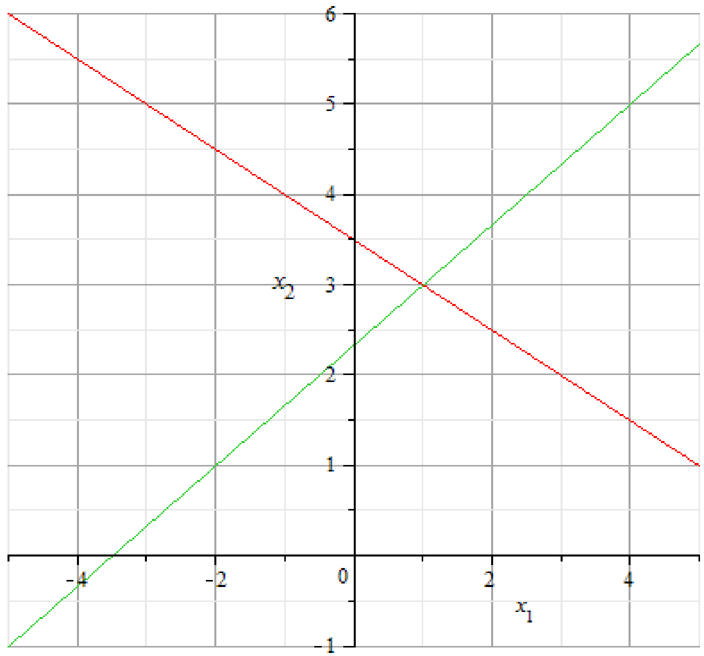
Solve both equations for x_2:

$$x_2 = -\frac{1}{2}x_1 + \frac{7}{2}$$
$$x_2 = \frac{2}{3}x_1 + \frac{7}{3}$$

GRAPHICAL METHOD (Cont.)

Solution (Cont.)

Plot the two equations on Cartesian coordinate:



The solution is given by the intersection point of the two lines, $x_1 = 1$ and $x_2 = 3$.

GRAPHICAL METHOD (Cont.)

Given the two lines L_1 and L_2, one and only one of the following may occur:

- If L_1 and L_2 intersect at exactly one point, there is a unique solution. The lines cross at only one point.
- If L_1 and L_2 are parallel and coincident, there is an infinitely many solutions.
- If L_1 and L_2 are almost parallel, the system is ill-conditioned. The point of intersection is difficult to detect visually due to the slopes are closed.
- If L_1 and L_2 are parallel and distinct, there is no solution.

GRAPHICAL METHOD (Cont.)

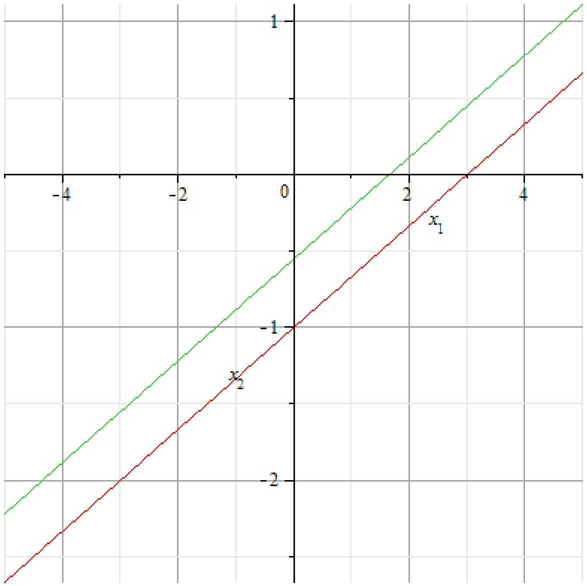
Example 2

Determine the nature of the solution for the following system of linear equations

$$x_1 - 3x_2 = 3$$

$$3x_1 - 9x_2 = 5$$

Solution



No solution

GRAPHICAL METHOD (Cont.)

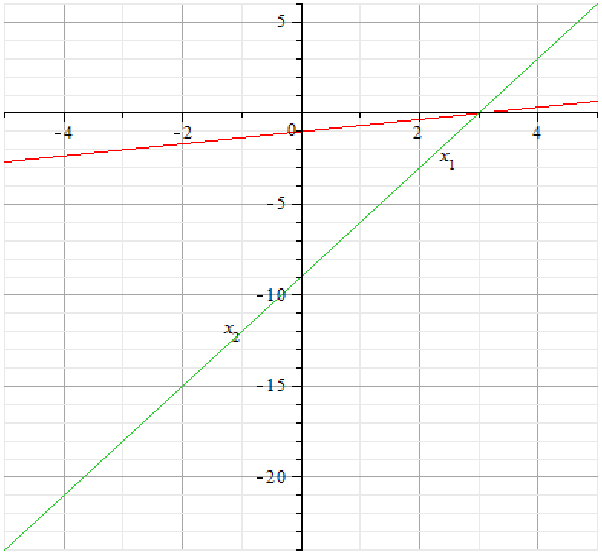
Example 3

Determine the nature of the solution for the following system of linear equations

$$x_1 - 3x_2 = 3$$

$$3x_1 - x_2 = 9$$

Solution



Unique solution

GRAPHICAL METHOD (Cont.)

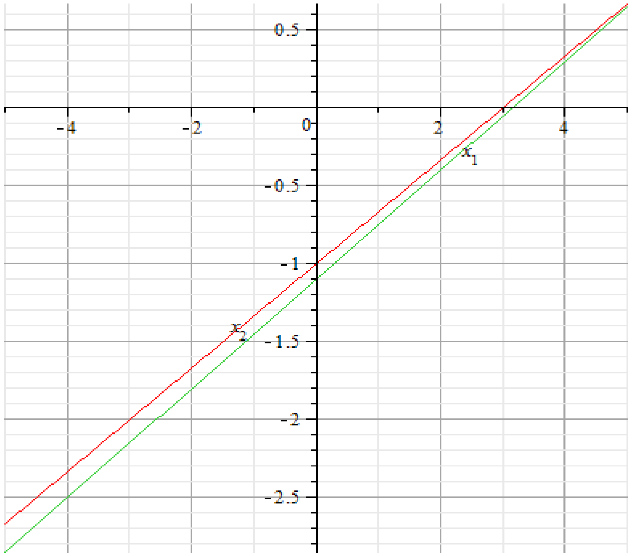
Example 4

Determine the nature of the solution for the following system of linear equations

$$x_1 - 3x_2 = 3$$

$$0.35x_1 - x_2 = 1.1015$$

Solution



ill-conditioned

GRAPHICAL METHOD (Cont.)

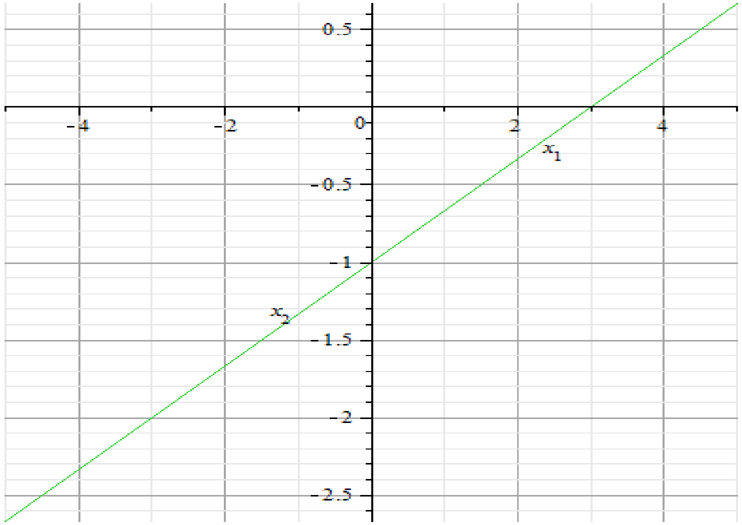
Example 5

Determine the nature of the solution for the following system of linear equations

$$x_1 - 3x_2 = 3$$

$$2x_1 - 6x_2 = 6$$

Solution



Many solutions

CRAMER'S RULE

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Definition

Cramer's rule states that each unknown in a system of linear equations may be expressed as a fraction of two determinants. The denominator, D is a determinant of $\boldsymbol{A}$ matrix. The numerator, D_n is obtained by replacing the column of the coefficients of the unknown by the vector $\boldsymbol{b}$.

Cramer's rule formula:

$$x_n = \frac{D_n}{D}$$

CRAMER'S RULE (Cont.)

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Determinant for small dimension of matrices

1×1	$\left\|a_{11}\right\| = a_{11}$
2×2	$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$
3×3	$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$
	$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

CRAMER'S RULE (Cont.)

For 3 × 3 matrices, the Cramer's rule are:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

CRAMER'S RULE (Cont.)

Example 5

Given the system of linear equations

$$x_1 + 3x_2 - x_3 = -8$$
$$2x_1 - x_2 + 3x_3 = 13$$
$$3x_1 + 2x_2 - x_3 = -4$$

a) Transform the above system into a matrix form of $\mathbf{Ax} = \mathbf{b}$.
b) Compute x_1, x_2 and x_3 by using Cramer's rule.

Solution

a)

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 13 \\ -4 \end{bmatrix}$$

CRAMER'S RULE (Cont.)

Solution (Cont.)

b) Find the determinant of the matrix **A**

$$D = |\mathbf{A}| = \begin{vmatrix} 1 & 3 & -1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{vmatrix} = (1)\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 21$$

Find the determinants of D_1, D_2 and D_3 by replacing the first, second and third column in **A** with **b**.

$$D_1 = \begin{vmatrix} -8 & 3 & -1 \\ 13 & -1 & 3 \\ -4 & 2 & -1 \end{vmatrix} = (-8)\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} 13 & 3 \\ -4 & -1 \end{vmatrix} + (-1)\begin{vmatrix} 13 & -1 \\ -4 & 2 \end{vmatrix} = 21$$

CRAMER'S RULE (Cont.)

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Solution (Cont.)

$$D_2 = \begin{vmatrix} 1 & -8 & -1 \\ 2 & 13 & 3 \\ 3 & -4 & -1 \end{vmatrix} = (1)\begin{vmatrix} 13 & 3 \\ -4 & -1 \end{vmatrix} - (-8)\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} + (-1)\begin{vmatrix} & 2 & 13 \\ 3 & & \\ & -4 & -4 \end{vmatrix} = -42$$

$$D_3 = \begin{vmatrix} 1 & 3 & -8 \\ 2 & -1 & 13 \\ 3 & 2 & -4 \end{vmatrix} = (1)\begin{vmatrix} -1 & 13 \\ 2 & -4 \end{vmatrix} - 3\begin{vmatrix} 2 & 13 \\ 3 & -4 \end{vmatrix} + (-8)\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 63$$

$$x_1 = \frac{21}{21} = 1, \quad x_2 = \frac{-42}{21} = -2, \quad x_3 = \frac{63}{21} = 3$$

Therefore, $x_1 = 1, x_2 = -2, \ x_3 = 3.$

Conclusion

- **Graphical method** is applicable for system with two equations only.
- **Cramer's rule** are suitable to solve 3×3 system.
- However, the difficulty arise in Cramer's rule when the dimension of the matrix is high as one needs to compute the **determinant of the matrix**.

Author Information

Mohd Zuki Salleh
Applied & Industrial Mathematics Research Group,
Faculty of Industrial Sciences & Technology (FIST),
Universiti Malaysia Pahang,
Lebuhraya Tun Razak,
26300 Gambang, Kuantan,
Pahang, Malaysia
Web: Mohd Zuki Salleh
UMPIR: Mohd Zuki Salleh
Google Scolar: Mohd Zuki Salleh
ResearchID: F-5440-2012
ORCiD: http://orcid.org/0000-0001-9732-9280
SCOPUS ID: 25227025600
E-mail: zuki@ump.edu.my

Norhayati Binti Rosli,
Senior Lecturer,
Applied & Industrial Mathematics Research Group,
Fakulti Sains & Teknologi Industri,
Universiti Malaysia Pahang,
26300 Gambang, Pahang.
SCOPUS ID: 36603244300
UMPIR ID: 3449
Google Scholars: https://scholar.google.com/citations?user=SLoPW9oAAAAJ&hl=en
e-mail: norhayati@ump.edu.my