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Numerical Methods
Roots of Equations

by

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Description

AIMS

This chapter is aimed to compute the root(s) of the equations by using graphical method and numerical methods.

EXPECTED OUTCOMES

1. Students should be able to find roots of the equations by using graphical approach and incremental search.
2. Students should be able to find the roots of the equations by using bracketing and open methods.
3. Students should be able to provide the comparison between bracketing and open methods.
4. Students should be able to calculate the approximate and true percent relative error.

REFERENCES

1. Norhayati Rosli, Nadirah Mohd Nasir, Mohd Zuki Salleh, Rozieana Khairuddin, Nurfatihah Mohamad Hanafi, Noraziah Adzhar. *Numerical Methods,* Second Edition, UMP, 2017 (Internal use)
2. Chapra, C. S. & Canale, R. P. *Numerical Methods for Engineers*, Sixth Edition, McGraw–Hill, 2010.

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5. Open Method
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   - Secant Method

INTRODUCTION

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- Mathematical model in science and engineering involve equations that need to be solved.
- Equation of one variable can be formulated as

$$f(x) = 0 \qquad (1)$$

- Equation (1) can be in the form of linear and nonlinear.
- Solving equation (1) means that finding the values of *x* that satisfying equation (1).

INTRODUCTION (Cont.)

Equation (1) may belong to one of the following types of equations

1 **Algebraic equations**

2 **Polynomial equations**

3 **Transcendental equations**

A non-algebraic equation of trigonometric, exponential and logarithm function

INTRODUCTION (Cont.)

Example 1: Algebraic Equation

$$4x - 3x^2y - 15 = 0$$

Example 2: Polynomial Equation

$$x^2 + 2x - 4 = 0$$

Example 3: Transcendental Equation

$$\sin(2x) - 3x = 0$$

INTRODUCTION (Cont.)

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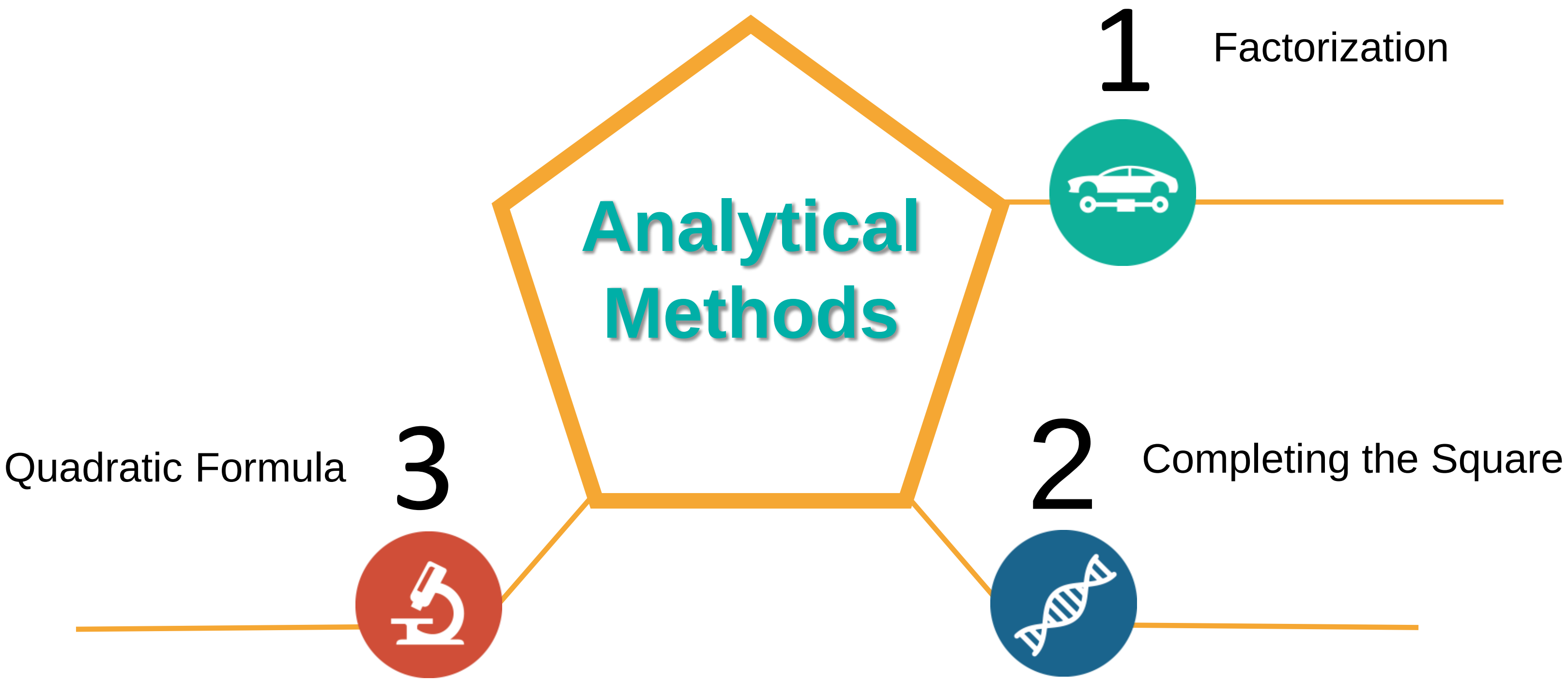
Finding Roots for Quadratic Equations

$$f(x) = ax^2 + bx + c$$



INTRODUCTION (Cont.)

- All above mentioned methods to solve quadratic equations are a**nalytical methods**
- The solution obtained by using analytical methods is called **exact solution**
- Due to the complexity of the equations in modelling the real life system, the exact solutions are often difficult to be found.
- Thus require the used of **numerical methods.**
- The solution that obtained by using numerical methods is called **numerical solution.**

INTRODUCTION (Cont.)

Three types of Numerical Methods shall be considered to find the roots of the equations:

Finding Roots using Numerical Methods

1 **Incremental Search**

2 **Bracketing Methods**

1. Bisection Method
2. False Position Method

3 **Open Methods**

1. Newton Raphson Method
2. Secant Method

Prior to the numerical methods, a **graphical method** of finding roots of the equations are presented.

GRAPHICAL METHOD

- Graphical method is the simplest method
- The given function is plotted on Cartesian coordinate and x −values (roots) that satisfying $f(x) = 0$ is identified.
- x −values (roots) satisfying $f(x) = 0$ provide approximation roots for the underlying equations.
- $f(x)$ can have one or possibly many root(s).

GRAPHICAL METHOD (Cont.)

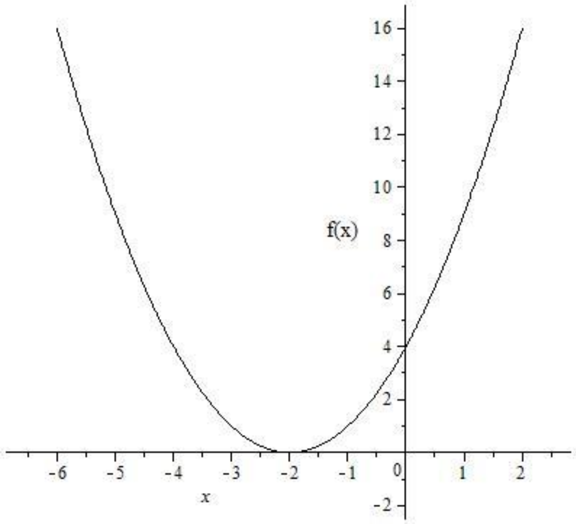


Figure 1 : One Solution

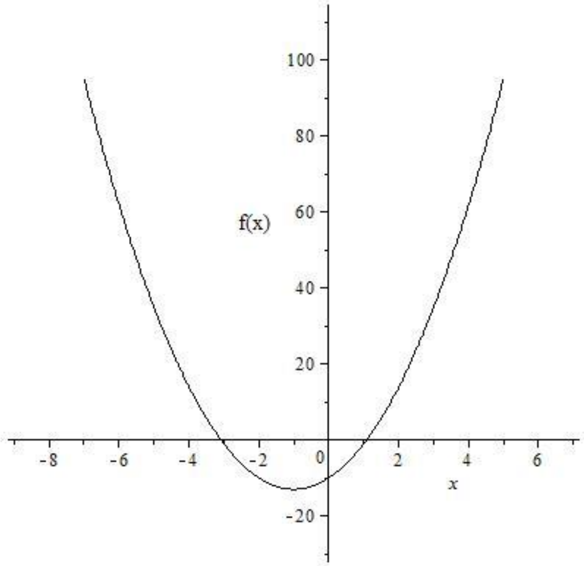


Figure 2 : Two Solutions

GRAPHICAL METHOD (Cont.)

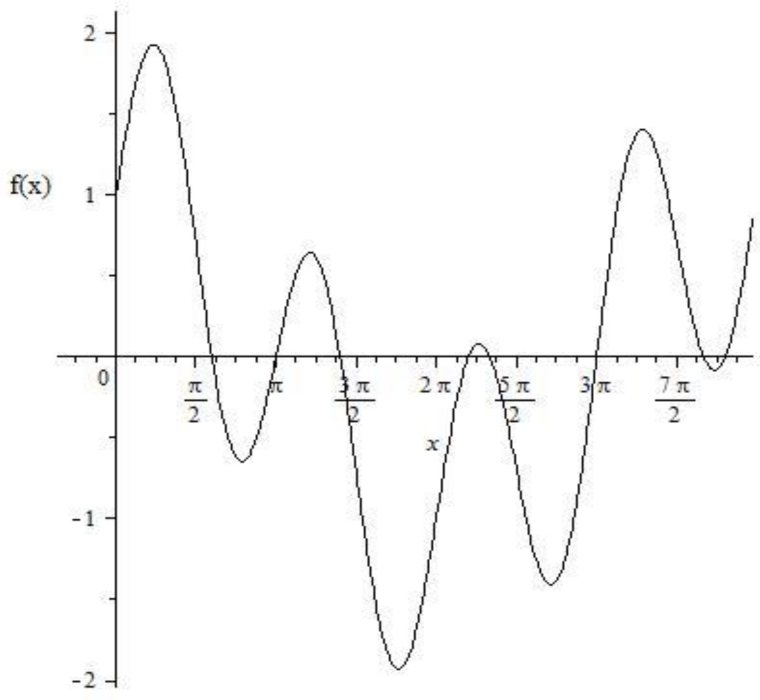


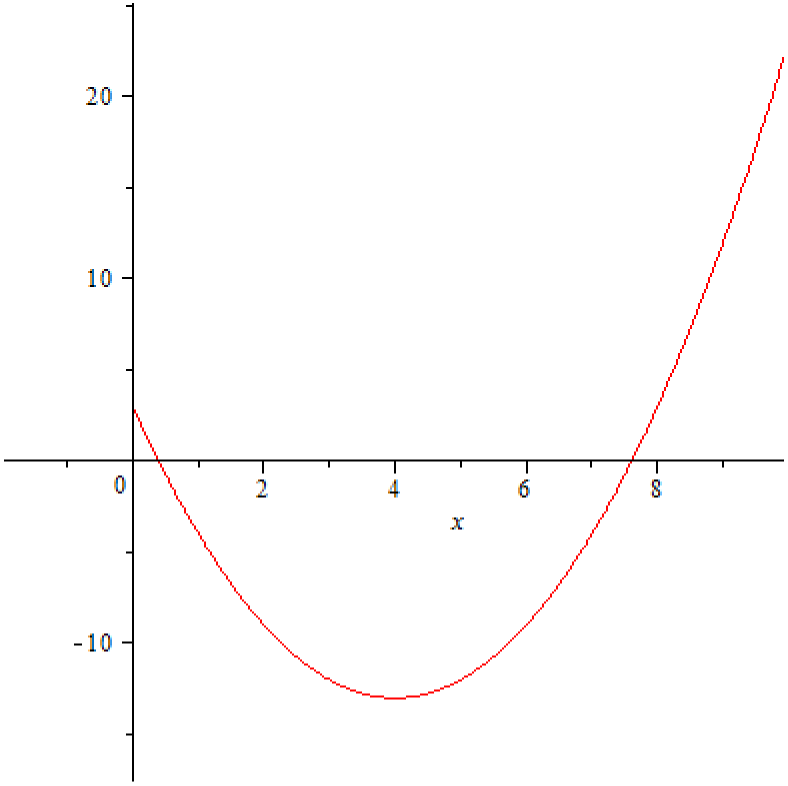
Figure 3 : Many Solutions

GRAPHICAL METHOD (Cont.)

Example 4

Find root(s) of $f(x) = x^2 - 8x + 3$ by using graphical method.

Solution



Based on the graph, the function $f(x)$ cross x –axis at two points.
Therefore there are two roots for $f(x)$
The approximate roots of $f(x)$ are 0.364 and 7.663

GRAPHICAL METHOD (Cont.)

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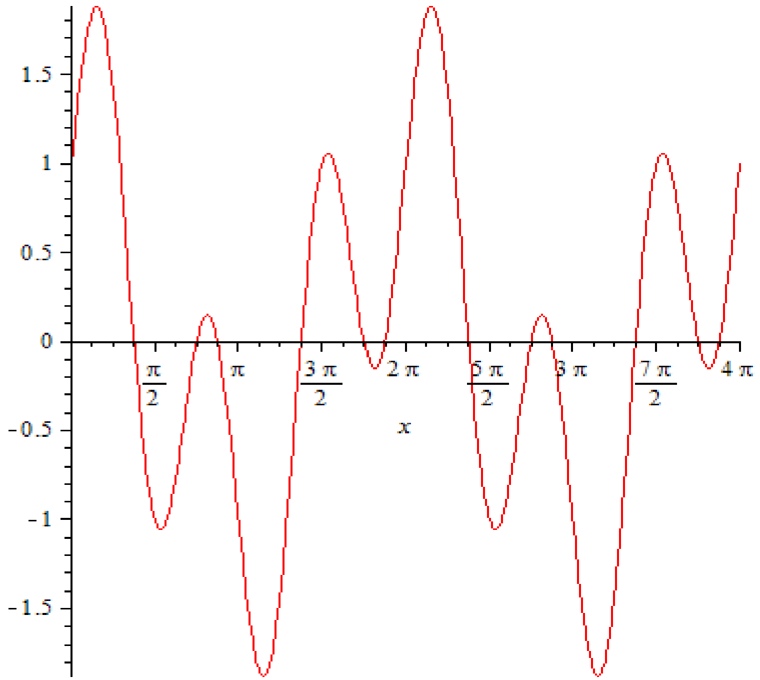
Example 5

Find root(s) of $f(x) = \cos(x) + \sin(3x)$ for $0 \leq x \leq 4\pi$ by using graphical method.

Solution



There are twelve roots for $f(x)$ since the function cross x −axis at twelve points. The approximate roots of $f(x)$ are 1.238, 2.401, 2.701, 4.239, 5.439, 5.852, 7.39, 8.628, 8.966, 10.691, 11.704 and 12.154

GRAPHICAL METHOD (Cont.)

Example 6 [Chapra & Canale]

The velocity of a free falling parachutist is given as

$$v = \frac{gm}{c}\left(1 - e^{-\left(c/m\right)t}\right)$$

Use the graphical approach to determine the drag coefficient, c needed for a parachutist of mass, $m = 68.1$ kg to have a velocity of $40\ \mathrm{m}s^{-1}$ after free falling for time, 10s. Given also gravity is $9.8\ \mathrm{m}s^{-2}$

Solution

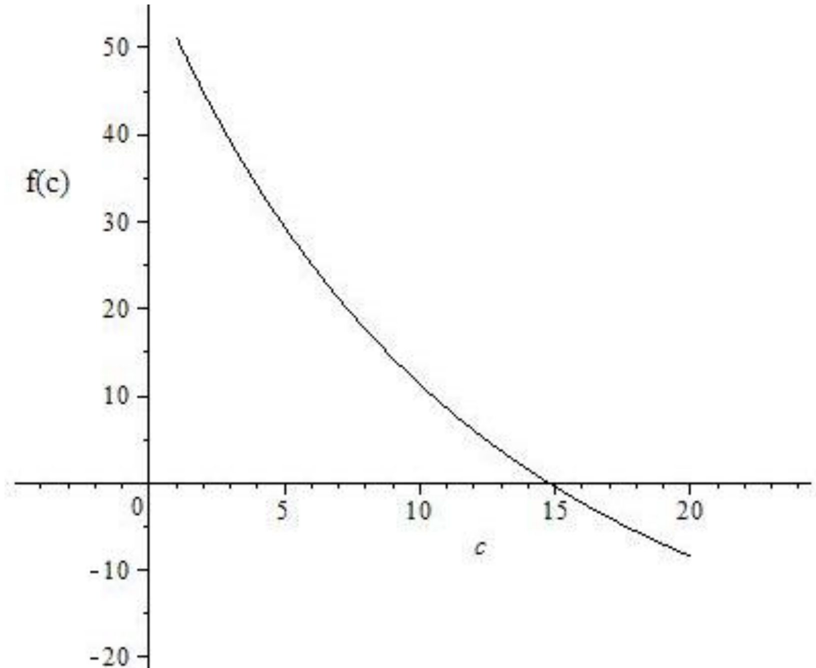
To determine the root of drag coefficient, c. we need to have a function $f(c) = 0$. Substituting the values given in the equation and rearranging yield

$$f(c) = \frac{9.8(68.1)}{c}\left(1 - e^{-\left(c/68.1\right)10}\right) - 40 = 0$$

GRAPHICAL METHOD (Cont.)

Solution (cont.)

Plot the function $f(c)$ and determine where the graph crosses the horizontal axis.



x	$f(x)$
4	34.115
8	17.653
12	6.0670
16	-2.2690
20	-8.4010

Functions have opposite sign

From the graphical view, the root exists between $c = 12$ and $c = 16$, where the functions $f(12)$ and $f(16)$ have opposite sign, that is $f(12) \times f(16) < 0$.

INCREMENTAL SEARCH

- Incremental search is a technique of calculating $f(x)$ for incremental values of x over the interval where the root lies.
- It starts with an initial value, x_0.
- The next value x_n for $n = 1,2,3, \ldots$ is calculated by using

  $$x_n = x_{n-1} + h$$

  where h is referred to a step size.
- If the sign of two $f(x)$ changes or if

  $$f(x_n) \cdot f(x_{n-1}) < 0$$

  then the root exist over the prescribed interval of the lower bound, x_l and upper bound, x_u.
- The root is estimated by using

  $$x_r = \frac{x_l + x_u}{2}$$

INCREMENTAL SEARCH (Cont.)

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Example 6

Find the first root of $f(x) = 4.15x^2 - 16x + 8$ by using incremental search. Start the procedure with the initial value, $x_0 = 0$ and the step size, $h = 0.1$. Perform three iterations of the incremental search to achieve the best approximation root.

Solution

Start the estimation with initial value $x_0 = 0$ and step size, $h = 0.1$.

x	$f(x)$
0	8
0.1	6.4415
0.2	4.966
0.3	3.5735
0.4	2.264
0.5	1.0375
0.6	-0.106

$$f(0.5) \cdot f(0.6) < 0$$

$$x_r = \frac{0.5 + 0.6}{2} = 0.55$$

INCREMENTAL SEARCH (Cont.)

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Solution (Cont.)

Increasing the accuracy of root estimation with step size, $h = 0.01$ for $x \in [0.5, 0.6]$

x	$f(x)$
0.5	1.0375
0.51	0.919415
0.52	0.80216
0.53	0.685735
0.54	0.57014
0.56	0.455375
0.57	0.34144
0.58	0.11606
0.59	0.004615
0.60	-0.106

$$f(0.59) \cdot f(0.6) < 0$$

$$x_r = \frac{0.59 + 0.6}{2} = 0.5950$$

INCREMENTAL SEARCH (Cont.)

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Solution (Cont.)

Increasing the accuracy of root estimation with step size, $h = 0.001$ for $x \in [0.59, 0.6]$

x	$f(x)$
0.59	0.004615
0.591	-0.0064385
0.592	-0.0175744
0.593	-0.02865665
0.594	-0.097306

$$f(0.59) \cdot f(0.591) < 0$$

$$x_r = \frac{0.59 + 0.591}{2} = 0.5905$$

$$\varepsilon_a = \left| \frac{0.5905 - 0.595}{0.5905} \right| \times 100\% = 0.76\%$$

For three iterations, the first root of $f(x) = 4.15x^2 - 16x + 8$ is 0.5905 with $\varepsilon_a = 0.76\%$

BRACKETING METHODS

- Figure 1 illustrates the basic idea of bracketing method–that is guessing an interval containing the root(s) of a function.
- Starting point of the interval is a lower bound, x_l. End point of the interval is an upper bound, x_u.
- By using bracketing methods, the interval will split into two subintervals and the size of the interval is successively reduced to a smaller interval.
- The subintervals will reduce the range of intervals until its distance is less than the desired accuracy of the solution

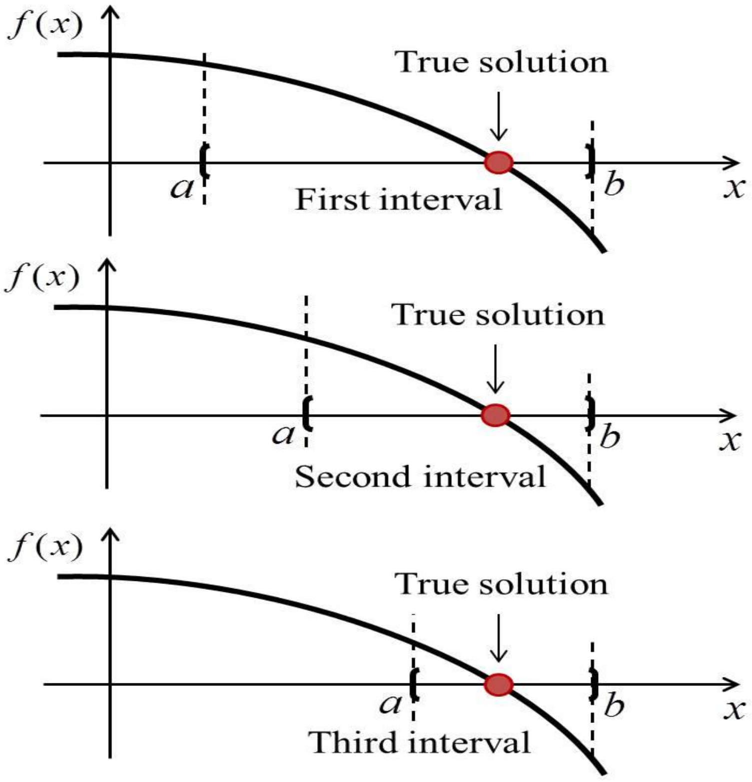
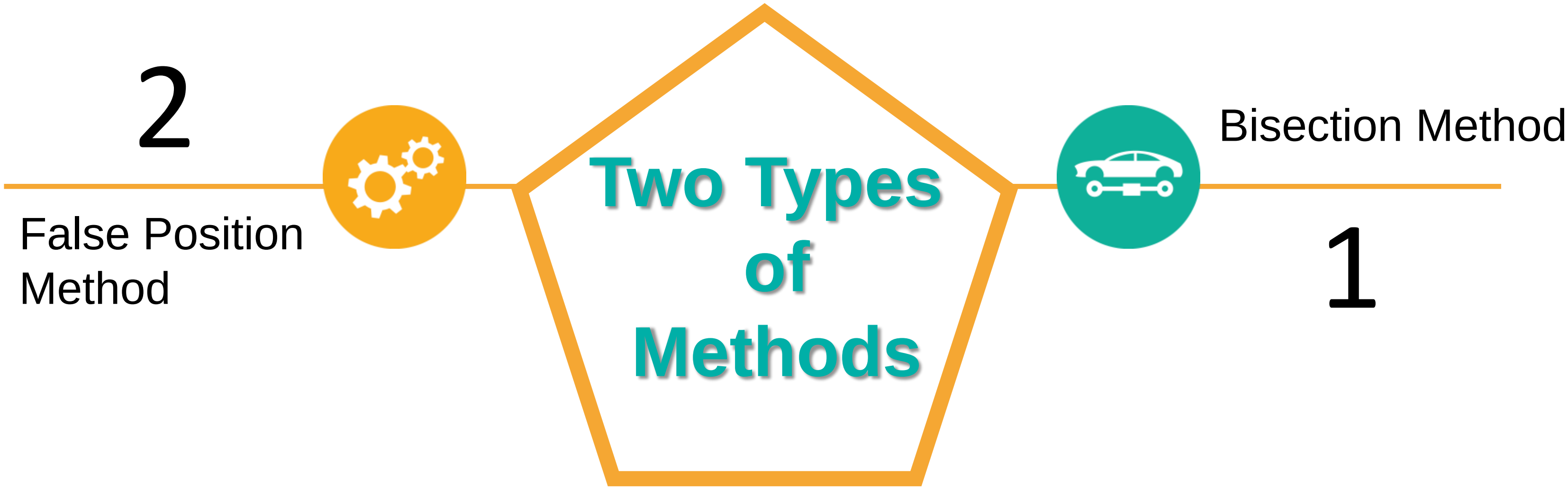


Figure 4: Graphical Illustration of Bracketing Method

BRACKETING METHODS

- Bracketing methods always converge to the true solution.
- There are two types bracketing methods; bisection method and false position method.



BISECTION METHOD

- Bisection method is the simplest bracketing method.
- The lower value, x_l and the upper value, x_u which bracket the root(s) are required.
- The procedure starts by finding the interval $[x_l, x_u]$ where the solution exist.
- As shown in **Figure 5**, at least one root exist in the interval $[x_l, x_u]$ if $f(x_l) \cdot f(x_u) < 0$

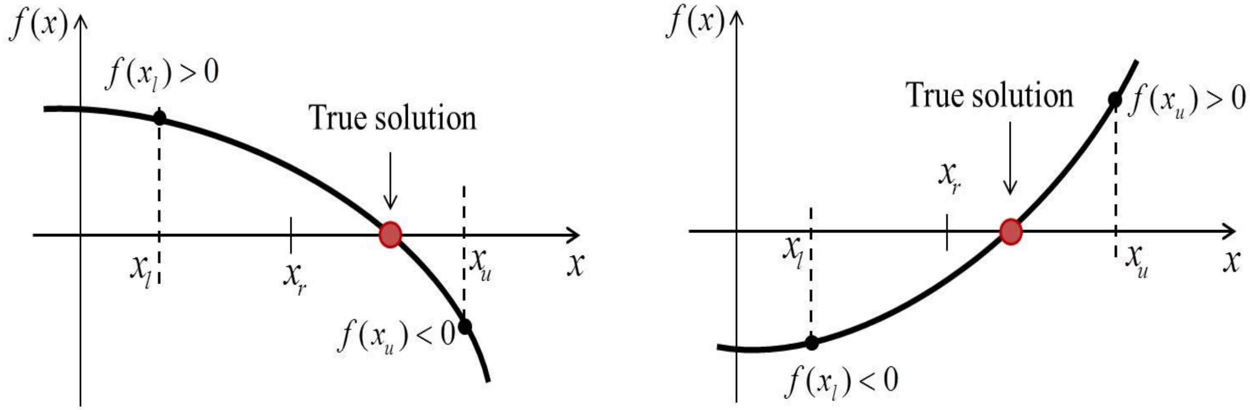


Figure 5: Solution of $f(x) = 0$

BISECTION METHOD (Cont.)

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Algorithm

For the continuous equation of one variable, $f(x) = 0$,

Step 1: Choose the lower guess, x_l and the upper guess, x_u that bracket the root such that the function has opposite sign over the interval, $x_l \leq x \leq x_u$.

Step 2: The estimation root, x_r is computed by using

$$x_r = \frac{x_l + x_u}{2}$$

Step 3: Use the following evaluations to identify the subinterval that the root lies

- ✓ If $f(x_l) \cdot f(x_r) < 0$, then the root lies in the lower subinterval. Therefore, set $x_u = x_r$ and repeat **Step 2.**
- ✓ If $f(x_l) \cdot f(x_r) > 0$, then the root lies in the upper subinterval. Therefore set $x_l = x_r$ and repeat **Step 2.**
- ✓ If $f(x_l) \cdot f(x_r) = 0$, then the root is equal to x_r. Terminate the computation.

Step 4: Calculate the approximate percent relative error,

$$\varepsilon_a = \left| \frac{x_r^{present} - x_r^{previous}}{x_r^{present}} \right| \times 100\%$$

Step 5: Compare with. If $\varepsilon_a < \varepsilon_s$, then stop the computation. Otherwise go to **Step 2** and repeat the process by using the new interval.

BISECTION METHOD (Cont.)

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Example 7

Use three iterations of the bisection method to determine the root of $f(x) = -0.6x^2 + 2.4x + 5.5$. Employ initial guesses, $x_l = 5$ and $x_u = 10$. Compute the approximate percent relative error, ε_a and true percent relative error, ε_t after each iteration.

Solution

Calculate the true value for the given quadratic function $f(x) = -0.6x^2 + 2.4x + 5.5$ using quadratic formula (or you can calculate directly by using the calculator)

$$x = \frac{-2.4 \pm \sqrt{(2.4)^2 - 4(-0.6)(5.5)}}{2(-0.6)}$$

$$x = -1.6286, \quad x = 5.6286$$

Choose the true value, $x = 5.6286$ for the highest root of $f(x)$. Estimate the root of $f(x)$ using bisection method with initial guess $x_l = 5$ and $x_u = 10$.

BISECTION METHOD (Cont.)

Solution (Cont.)

Estimate the root of $f(x)$ using bisection method with initial interval $[5,10]$.

- First iteration, $x \in [5,10]$

$$f(5) = 2.50$$

$$f(10) = -30.50$$

First estimate using bisection method formula

$$x_r = \frac{5+10}{2} = 7.5$$

$$f(7.5) = -10.25$$

Since $f(x_l) \cdot f(x_r) < 0$, the root lies in the lower subinterval. Then set $x_u = 7.5$.

$$\varepsilon_t = \left|\frac{5.6286 - 7.5}{5.6286}\right| \times 100\% = 33.23\% \text{ and } \varepsilon_a = -$$

BISECTION METHOD (Cont.)

Solution (Cont.)

- Second iteration, $x \in [7.5,10]$

$$f(5) = -10.25$$

$$f(10) = -30.50$$

First estimate using bisection method formula

$$x_r = \frac{7.5+10}{2} = 6.25$$

$$f(6.25) = -2.9375$$

Since $f(x_l) \cdot f(x_r) < 0$, the root lies in the lower subinterval. Then set $x_u = 6.25$.

$$\varepsilon_t = \left|\frac{5.6286-6.25}{5.6286}\right| \times 100\% = 11.04\% \text{ and } \varepsilon_a = 20\%$$

BISECTION METHOD (Cont.)

Solution (Cont.)

Continue the third iteration for $x \in [5, 6.25]$. The results are summarized in the following table.

i	x_l	x_u	x_r	$f(x_l)$	$f(x_u)$	$f(x_r)$	$f(x_l) \cdot f(x_r)$	ε_t	ε_a
1	5	10	7.5	2.5	-30.50	-10.25	-25.625	33.25	-
2	5	7.5	6.25	2.5	-10.25	-2.9375	-7.3438	11.04	20.00
3	5	6.25	5.625	2.5	-2.9375	0.0156	-0.0391	0.06	11.11

Therefore, after three iterations the approximate root of $f(x)$ is $x_r = 5.6250$ with $\varepsilon_t = 0.06\%$ and $\varepsilon_a = 11.11\%$.

FALSE POSITION METHOD (Cont.)

- It is an improvement of the Bisection method.
- The bisection method converges slowly due to its behavior in redefined the size of interval that containing the root.
- The procedure begins by finding an initial interval $[x_l, x_u]$ that bracket the root.
- $f(x_l)$ and $f(x_u)$ are then connected using a straight line.
- The estimated root, x_r is the x-value where the straight line crosses x-axis.
- **Figure 6** indicates the graphical illustration of False Position method.

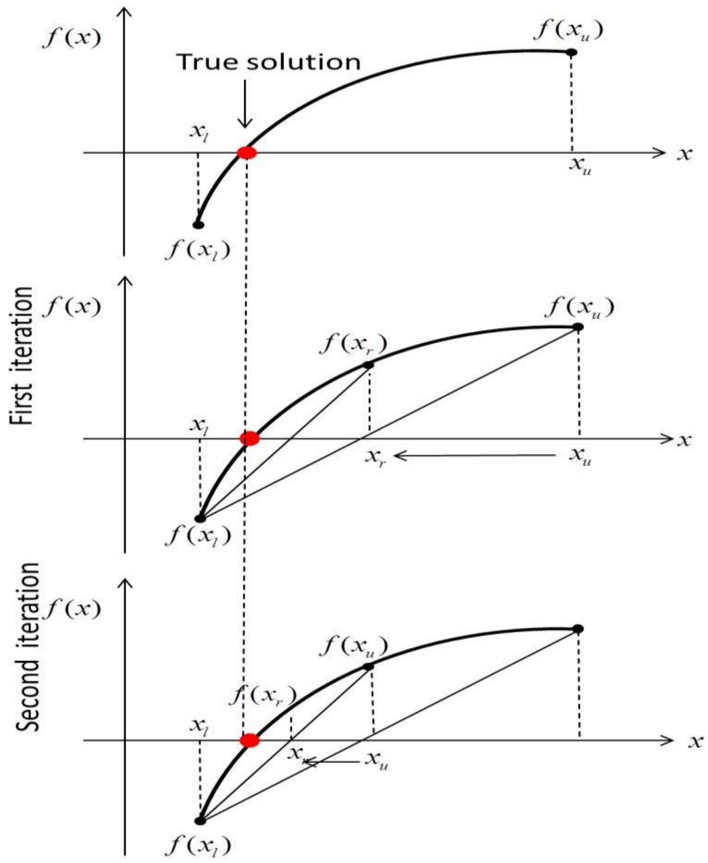


Figure 6: Graphical Illustration of False Position Method

FALSE POSITION METHOD (Cont.)

False Position Method Formula

Straight line joining the two points $(x_l, f(x_l))$ and $(x_u, f(x_u))$ is given by

$$\frac{f(x_u)-f(x_l)}{x_u-x_l}=\frac{y-f(x_u)}{x-x_u}$$

Since the line intersect the x-axis at x_r, so for $x=x_r, y=0$, the following is obtained

$$x_r-x_u=-\frac{f(x_u)(x_u-x_l)}{f(x_u)-f(x_l)}$$

Rearranging the second equation yields the **False Position Method Formula**

$$x_r=x_u-\left[\frac{f(x_u)(x_l-x_u)}{f(x_l)-f(x_u)}\right]$$

FALSE POSITION METHOD (Cont.)

Algorithm

For the continuous equation of one variable, $f(x) = 0$,

Step 1: Choose the lower guess, x_l and the upper guess, x_u that bracket the root such that the function has opposite sign over the interval, $x_l \le x \le x_u$.

Step 2: The estimation root, x_r is computed by using

$$x_r = x_u - \left[\frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \right]$$

Step 3: Use the following evaluations to identify the subinterval that the root lies

- ✓ If $f(x_l) \cdot f(x_r) < 0$, then the root lies in the lower subinterval. Therefore, set $x_u = x_r$ and repeat **Step 2.**
- ✓ If $f(x_l) \cdot f(x_r) > 0$, then the root lies in the upper subinterval. Therefore set $x_l = x_r$ and repeat **Step 2.**
- ✓ If $f(x_l) \cdot f(x_r) = 0$, then the root is equal to x_r. Terminate the computation.

Step 4: Calculate the approximate percent relative error,

$$\varepsilon_a = \left| \frac{x_r^{present} - x_r^{previous}}{x_r^{present}} \right| \times 100\%$$

Step 5: Compare with. If $\varepsilon_a < \varepsilon_s$, then stop the computation. Otherwise go to **Step 2** and repeat the process by using the new interval.

FALSE POSITION METHOD (Cont.)

Example 8

Determine the first root $f(x) = -3x^3 + 19x^2 - 20x - 13$ by using False position method. Use the initial guesses of $x_l = -1$ and $x_u = 0$ with stopping criterion, ε_s=1%.

Solution

- First iteration, $x \in [-1,0]$

$$f(-1) = 29$$

$$f(0) = -13$$

First estimate using False position method is

$$x_r = 0 - \frac{(-13)(-1-0)}{29-(-13)} = -0.3095$$

$$f(-0.3095) = -4.9010$$

Since $f(x_l) \cdot f(x_r) < 0$, the root lies in the lower subinterval. Then set $x_u = -0.3095$.

$$\varepsilon_a = -$$

FALSE POSITION METHOD (Cont.)

Solution

- Second iteration, $x \in [-1, -0.3095]$.

Second estimate is

$$x_r = -0.3095 - \frac{(-4.9010)(-1+0.3095)}{29-(-4.9010)} = -0.4093$$

$$f(-0.4093) = -1.4253$$

Since $f(x_l) \cdot f(x_r) < 0$, the root lies in the lower subinterval. Then set $x_u = -0.4093$.

$$\varepsilon_a = 24.38\%$$

FALSE POSITION METHOD (Cont.)

Solution (Cont.)

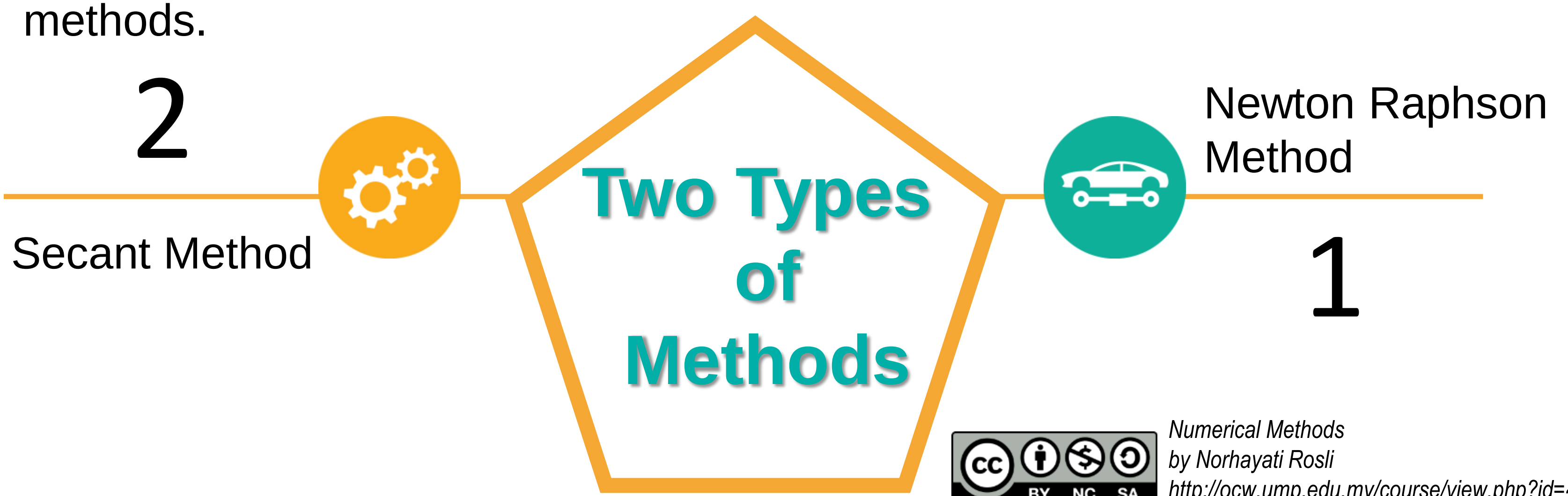
Continue the third iteration for $x \in [-1, -0.4093]$. The results are summarized in the following table.

i	x_l	x_u	x_r	$f(x_l)$	$f(x_u)$	$f(x_r)$	$f(x_l) \cdot f(x_r)$	ε_a
1	-1	0	-0.3095	29	-13	-4.0910	-142.1290	-
2	-1	-0.3095	-0.4093	29	-4.9003	-1.4253	-41.3337	24.38
3	-1	-0.4093	-0.4370	29	-1.4241	-0.3812	-11.0548	6.33
4	-1	-0.4370	-0.4443	29	-0.3820	-0.1002	-2.0907	1.65
5	-1	-0.4443	-0.4462	29	-0.1002	-0.0267	-0.7743	0.43

Therefore, after fifth iterations the approximate root of $f(x)$ is $x_r = -0.4462$ with $\varepsilon_a = 0.43\%$.

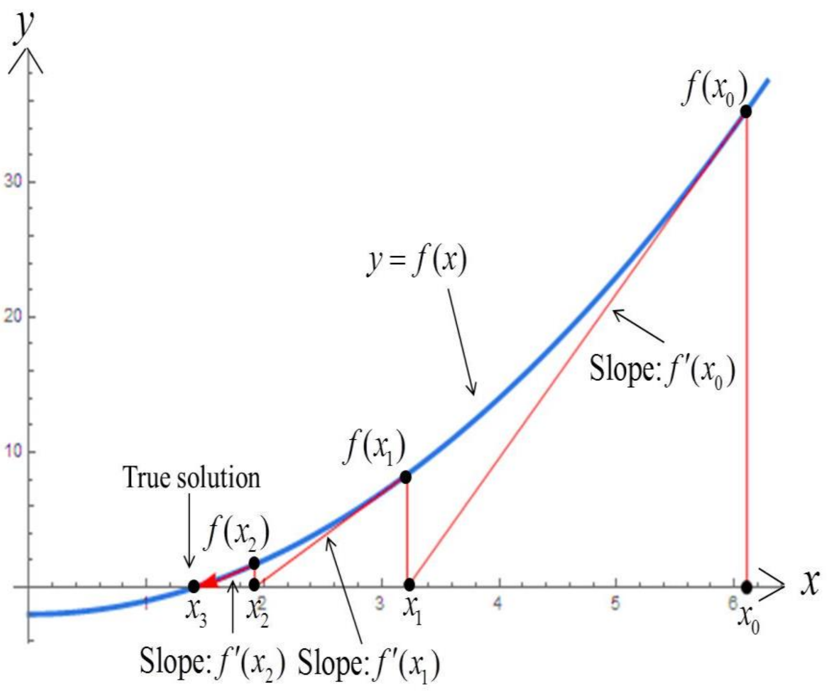
OPEN METHODS

- The idea of this method is to consider at least one initial guess which is not necessarily bracket the root.
- Normally, the chosen initial value(s) must be close to the actual root that can be found by plotting the given function against its independent variable.
- In every step of root improvement, x_r of previous step is considered as the previous value for the present step.
- In general, open methods provides no guarantee of convergence to the true value, but once it is converge, it will converge faster than bracketing methods.



NEWTON RAPHSON METHOD

- It is an open method for finding roots of $f(x) = 0$ by using the successive slope of the tangent line.
- The Newton Raphson method is applicable if $f(x)$ is continuous and differentiable.
- **Figure 6** shows the graphical illustration of Newton Raphson method
- Numerical scheme starts by choosing the initial point, x_0 as the first estimation of the solution.
- The improvement of the estimation of x_1 is obtained by taking the tangent line to $f(x)$ at the point $(x_0, f(x_0))$ and extrapolate the tangent line to find the point of intersection with an x–axis.



NEWTON RAPHSON METHOD (Cont.)

Slope for the first iteration is:

$$f'(x_0) = \frac{f(x_0) - 0}{x_0 - x_1} \quad (1)$$

Rearranging equation (1) yields:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- The next estimation, x_2 is the intersection of the tangent line $f(x)$ at the point $(x_1, f(x_1))$.
- The estimation, x_{i+1} is the intersection of the tangent line $f(x)$ at the point $(x_i, f(x_i))$. The slope of the i^{th} iteration is

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}} \quad (2)$$

Rearranging equation (2) gives **Newton Raphson Formula**:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

NEWTON RAPHSON METHOD (Cont.)

Algorithm

For the continuous and differentiable function, $f(x) = 0$:

Step 1: Choose initial value, x_0 and find $f'(x_0)$.

Step 2: Compute the next estimate, x_{i+1} by using Newton Raphson formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Step 3: Calculate the approximate percent relative error, ε_a

$$\varepsilon_a = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100\%$$

Step 4: Compare ε_s with ε_a. If $\varepsilon_a < \varepsilon_s$, the computation is stopped. Otherwise, repeat **Step 2**.

NEWTON RAPHSON METHOD (Cont.)

Example 8

Determine the first root $f(x) = 8e^{-x}\sin(x) - 1$ by using Newton Rapshon method. Use the initial guesses of $x_0 = 0.3$ and perform the computation up to three iterations.
(*Use radian mode in your calculator*)

Solution

Step 1

$$f(x) = 8e^{-x}\sin(x) - 1$$

$$f'(x) = 8e^{-x}\left(\cos(x) - \sin(x)\right)$$

Step 2

First iteration, $x_0 = 0.3$

$f(0.3) = 8e^{-0.3}\sin(0.3) - 1 = 0.7514,$

$f'(0.3) = 8e^{-0.3}(\cos(0.3) - \sin(0.3)) = 3.9104,$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.3 - \frac{0.7514}{3.9104} = 0.1078$$

$$\varepsilon_a = \left|\frac{0.1078 - 0.3}{0.1078}\right| \times 100\% = 178.18\%$$

NEWTON RAPHSON METHOD (Cont.)

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Solution (Cont.)

Continue the second iteration and the results are summarised as follows.

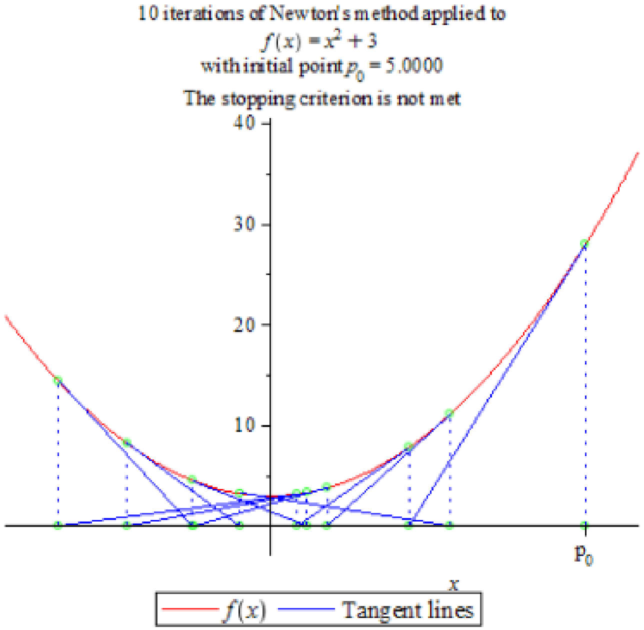
No. of iteration	i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}	ε_a (%)
1	0	0.3	0.7514	3.9104	0.1078	178.18
2	1	0.1078	-0.2270	6.3674	0.1435	24.84
3	2	0.1435	-0.0090	5.8684	0.1450	1.05

Therefore, after three iterations the approximated root of $f(x)$ is $x_3 = 0.1450$ with $\varepsilon_a = 1.05\%$.

NEWTON RAPHSON METHOD (Cont.)

Pitfalls of the Newton Raphson Method

Case 1: The tendency of the results obtained from the Newton Raphson method to oscillate around the local maximum or minimum without converge to the actual root.

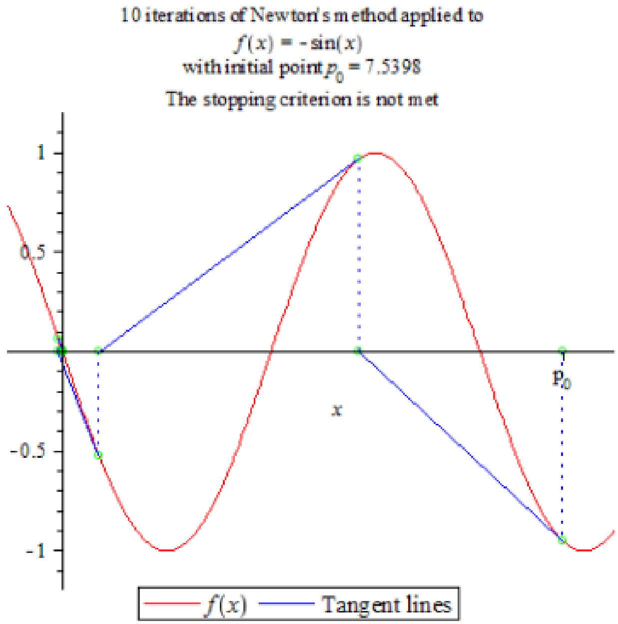


Case 2: Division by zero involve in the Newton Raphson formula when $f'(x) = 0$.

NEWTON RAPHSON METHOD (Cont.)

Pitfalls of the Newton Raphson Method

Case 3: In some cases where the function $f(x)$ is oscillating and has a number of roots, one may choose an initial guess close to a root. The guesses may jump and converge to some other roots and the process become oscillatory, which leads to endless cycle of fluctuations between x_i and x_{i+1} without converge to the desired root.



SECANT METHOD

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Introduction

- In many cases, the derivative of a function is very difficult to find or even is not differentiable.
- Alternative approach is by using secant method.

- The slope in Newton's Rapshon method is substituted with backward finite divided difference

$$f'(x_i) = \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

- The secant method formula is:

$$x_{i+1} = x_i - \left[\frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)} \right]$$

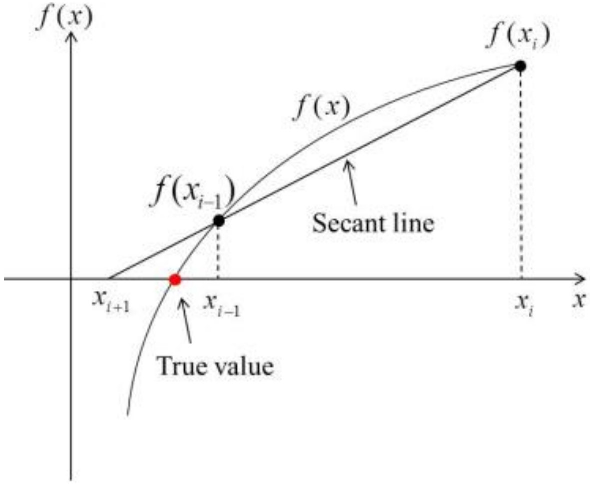

Figure 7: Graphical Illustration of Secant Method

SECANT METHOD (Cont.)

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Algorithm

For the continuous function, $f(x) = 0$:

Step 1: Choose initial values, x_{-1} and x_0. Find $f(x_{-1})$ and $f(x_0)$.

Step 2: Compute the next estimate, x_{i+1} by using secant method formula

$$x_{i+1} = x_i - \left[\frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)} \right]$$

Step 3: Calculate the approximate percent relative error, ε_a

$$\varepsilon_a = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100\%$$

Step 4: Compare ε_s with ε_a. If $\varepsilon_a < \varepsilon_s$, the computation is stopped. Otherwise, repeat **Step 2**.

SECANT METHOD (Cont.)

Example 9

Determine one of the real root(s) of $f(x) = -12 - 21x + 18x^2 - 2.4x^3$ by using secant method with initial guesses of $x_{-1} = 1.0$ and $x_0 = 1.3$. Perform the computation until $\varepsilon_a < 5\%$.

Solution

First iteration, $x_{-1} = 1.0$ and $x_0 = 1.3$

$f(1.0)$ =-17.4

$f(1.3) = -14.528$

$$x_1 = x_0 - \left[\frac{f(x_0)(x_{-1} - x_0)}{f(x_{-1}) - f(x_0)}\right]$$

$$= 1.3 - \left[\frac{-14.1528(1-1.3)}{-17.4+14.1528}\right] = 2.6075$$

$$\varepsilon_a = \left|\frac{2.6075-1.3}{2.6075}\right| \times 100\% = 50.14\% > \varepsilon_s$$

SECANT METHOD (Cont.)

Solution (Cont.)

Continue the second iteration and the results are summarised as follows.

No. of Iteration	i	x_{i-1}	x_i	$f(x_{i-1})$	$f(x_i)$	x_{i+1}	$\varepsilon_a(\%)$
1	0	1	1.3	-17.4	-14.1527	2.6075	50.14
2	1	1.3	2.6075	-14.1528	13.0780	1.9796	31.72
3	2	2.6075	1.9796	13.0780	-1.6519	2.0500	3.44

Therefore, after three iterations the approximated root of $f(x)$ is $x_3 = 2.0500$ with $\varepsilon_a = 3.44\%$.

Conclusion

Bracketing Method	Open Method
Need two initial guesses	Can involve one or more initial guesses
The root is located within an interval prescribed by a lower and an upper bound.	Not necessarily bracket the root.
Always work but converge slowly	Do not always work (can diverge) but when they do they usually converge much more quickly.

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