Numerical Methods
Introduction to Numerical Methods

By

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Description

AIMS

This chapter is an introduction to the numerical methods. It is aimed to:

1. introduce the concept of precision and accuracy.
2. compute the approximate and true percent relative errors.
3. approximate the functions by using Taylor's series.
4. define the concept of round-off error, truncation error and total numerical error.

EXPECTED OUTCOMES

1. Students should be able to define the concept of accuracy and precision.
2. Students should be able to compute the approximate percent relative error and the true percent relative error.
3. Students should be able to approximate the values of the functions by using Taylor's series expansion.
4. Students should be able to define two types errors involve in numerical methods.

REFERENCES

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2. Chapra, C. S. & Canale, R. P. *Numerical Methods for Engineers*, Sixth Edition, McGraw–Hill, 2010.

Content

1. What are Numerical Methods?
2. Errors
3. Taylor's Series Expansion
4. Truncation Error
5. Round-Off Error
6. Total Numerical Error

WHAT ARE NUMERICAL METHODS?

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Numerical Methods

1. The approximate methods for solving a wide variety of mathematical problems.

2. The solutions that obtained are called the approximate solutions.

3. It capable to handle nonlinearities, complex geometries and large system of coupled equations of many real physical situations that are difficult to be solved analytically

4. It can be implemented directly on digital computers

ERRORS

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Error Definition

1 Numerical error occurs during the computational process.

2 Error measure how close the approximate solution to the true or exact solution

3 It can be characterized by two types of measurements- accuracy and precision

ERRORS (Cont.)

Definition Accuracy & Precision

Accuracy & Precision

1 **Accuracy**
How close the computed or measured value to the true value

2 **Inaccuracy**
A systematic deviation from the actual value

3 **Precision**
How close a computed value to previously computed value

4 **Imprecision**
Magnitude of scatter

ERRORS (CONT.)

The relationship between the **True or Exact Value** and the **Approximate Value** can be formulated as:

$$\text{True Value} = \text{Approximate Value} + \text{Numerical Error} \qquad (1)$$

By rearranging, equation (1), the numerical error, $E(t)$ can be written as:

$$E(t) = \text{True Value - Approximate Value}$$

ERRORS (Cont.)

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True percent relative error, ε_t

$$\varepsilon_t = \left| \frac{\text{True value} - \text{Approximate Value}}{\text{True value}} \right| \times 100\%$$

Approximate percent relative error, ε_a

$$\varepsilon_a = \left| \frac{\text{Present Solution} - \text{Previous Solution}}{\text{Present Solution}} \right| \times 100\%$$

ERRORS (Cont.)

The iterative process is computed until

$$|\varepsilon_a| < \varepsilon_s \qquad (2)$$

where ε_s is a prespecified percent tolerance.

- ✓ If the relationships (2) is hold, then the acceptable result has been reached and no more iteration is required.
- ✓ The numerical result is assumed to be within the prespecified acceptable level.

ERRORS (Cont.)

Example 1

A civil engineer has measured the height of a 15 floor building as 3940 m and the working height of each beam as 35 m while the true values are 3945 m and 40 m, respectively. Compare their true error and true percent relative error.

Solution

True error in measuring the height of the building, $E_{t,1}$

$$E_{t,1} = 3945 - 3940 = 5\text{m}$$

True percent relative error in measuring the height of the building, $\varepsilon_{t,1}$

$$\varepsilon_{t,1} = \left| \frac{5}{3945} \right| \times 100\% = 0.13\%$$

ERRORS (Cont.)

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Solution (Cont.)

True error in measuring the height of the beam

$$E_{t,2} = 40 - 35 = 5\text{m}$$

True percent relative error in measuring the height of the beam

$$\varepsilon_{t,2} = \left|\frac{5}{40}\right| \times 100\% = 13\%$$

True percent relative error gives better measurement of the error compare than true error

ERRORS (Cont.)

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Example 2: Error Estimate for Iterative Methods

Taylor's series expansion of $\sin(x)$ is given by

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

Estimate $\sin\left(\frac{\pi}{3}\right)$ by adding terms one at a time to the above series. Iterate until $\varepsilon_a < 0.05\%$. If given the true value of $\sin\left(\frac{\pi}{3}\right) = 0.8660$, compute the true percent relative error, ε_t after each iteration.

Solution

i	Approximate value	ε_a	ε_t
1	$\sin\left(\frac{\pi}{3}\right) = \frac{\pi}{3} = 1.0472$	-	20.92%
2	$\sin\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \frac{\frac{\pi}{3}}{3!} = 0.8558$	22.37%	0.045%
3	$\sin\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \frac{\frac{\pi}{3}}{3!} + \frac{\left(\frac{\pi}{3}\right)^5}{5!} = 0.8663$	1.21%	0.04%
4	0.8660	0.04%	0%

$\varepsilon_a < 0.05\%$, hence stop the computation

TAYLOR'S SERIES EXPANSION

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Taylor's Series

1 Taylor series is used to approximate the value of the known function

2 Function is represented by a sum of terms of a convergent series

3 If only a few terms are used, the value of the function that is obtained from the Taylor series is an approximation value

4 Taylor series expansion of functions is used extensively in deriving numerical methods

TAYLOR'S SERIES EXPANSION (Cont.)

- A one dimensional Taylor series is a Taylor's expansion for a function $f(x)$ that is differentiable $(n+1)$ times in an interval containing a point $x = 0$.

- Taylor's theorem state that for each x in the interval, there exist a value $x = \delta$ in between $x = x_0$ and x.

Taylor's series expansion of order n is given by the formula

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f^{(3)}(x_0)(x - x_0)^3$$

$$+ \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + R_n(x)$$

where $R_n(x)$ is the remainder term.

TAYLOR'S SERIES EXPANSION (Cont.)

- The accuracy increase as x is closer to x_0 and as more terms are added in the series.
- The expansion of the Taylor's series around $x_0 = 0$ is called **Maclaurin series.**

Examples of the Taylor's series expansion for the selected functions are given in **Table 1**.

Table 1: Taylor's Series Expansions for the Selected Functions

$f(x)$	Taylor Series Expansion of $f(x)$
$\exp(x)$	$\exp(x_0)\left[1 + (x - x_0) + \frac{1}{2!}(x - x_0)^2 + \frac{1}{3!}(x - x_0)^3 + \cdots\right]$
$\cos(x)$	$\cos(x_0) - \sin(x_0)(x - x_0) - \frac{1}{2}\cos(x_0)\,(x - x_0)^2 + \frac{1}{6}\sin(x_0)\,(x - x_0)^3 + \cdots$
$\sin(x)$	$sin(x_0) + cos(x_0)(x - x_0) - \frac{1}{2}sin(x_0)\,(x - x_0)^2 - \frac{1}{6}\cos(x_0)\,(x - x_0)^3 + \cdots$

TAYLOR'S SERIES EXPANSION (Cont.)

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Example 3

Approximate the function $f(x) = \cos(x)$ by using Taylor series expansion about $x = 0$ using two, four and six terms. In each case calculate the approximate value of the function at $x = \frac{\pi}{12}$ and $x = \frac{\pi}{3}$.

Solution

Find the first five derivatives of the function and the value at $x = 0$ of the respective derivatives are substituted.

Step 1

n	Derivative of the Function	Value of the Derivative
0	$y(x) = \cos(x)$	$y(0) = 1$
1	$y'(x) = -\sin(x)$	$y'(0) = 0$
2	$y''(x) = -\cos(x)$	$y''(0) = -1$
3	$y^{(3)}(x) = \sin(x)$	$y^{(3)}(0) = 0$
4	$y^{(4)}(x) = \cos(x)$	$y^{(4)}(0) = 1$
5	$y^{(5)}(x) = -\sin(x)$	$y^{(5)}(0) = 0$

TAYLOR'S SERIES EXPANSION (Cont.)

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Solution (Cont.)

Step 2

Expanding the function in a Taylor series form and substituting the information in Step 1 in Taylor series

$$\cos(x) = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(0)$$

$$= 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0$$

TAYLOR'S SERIES EXPANSION (Cont.)

Solution (Cont.)

Step 3

The approximate values using two, four and six terms of Taylor series expansion are summarised in the following table.

Number of Terms	The Approximate Values for $x = \frac{\pi}{12}$	The Approximate Values for $x = \frac{\pi}{3}$
Two terms	$cos\left(\frac{\pi}{12}\right) = 1 + 0 = 1$	$cos\left(\frac{\pi}{3}\right) = 1 + 0 = 1$
Four Terms	$cos\left(\frac{\pi}{12}\right) = 1 - \frac{\left(\frac{\pi}{12}\right)^2}{2} = 0.9657305$	$cos\left(\frac{\pi}{3}\right) = 1 - \frac{\left(\frac{\pi}{3}\right)^2}{2} = 0.4516886$
Six Terms	0.9659263	0.5017962

TRUNCATION ERROR

- Truncation error resulting from the truncation of the numerical process in order to estimate finite number of terms from the sum of an infinite series.
- It arises due to the approximation method is used instead of the exact procedure.
- As in Example 2, the numerical evaluation of sine function by using Taylor's series is terminated after a certain term. This contribute to the truncation error.

ROUND-OFF ERROR

- Computer approximation retain only a fixed number of significant figures during a calculation.
- Numbers such as $\frac{1}{3} = 0.333333\ldots$, $e = 2.71828182\ldots$ and $\pi = 3.1415926\ldots$ cannot be expressed by a fixed number of significant figures.
- Therefore they cannot be represented exactly by the computer and must be round–off.
- There are two situations in numerical computations where a number can be shortened and lead to round off–errors which is either by chopping off or by rounding.
- Round off–errors can be reduced by increasing the numbers of significant figures

Example 4

1. $\frac{2}{3} = 0.6666$ if the actual number has been chopped off
2. $\frac{2}{3} = 0.6667$ if the actual number has been rounded

TOTAL NUMERICAL ERROR

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Total Numerical Error

1

Includes round–off and truncation errors

2

Sum of the round–off and truncation errors yield the total of numerical error

Conclusion

- Numerical method is an approximated method.
- The solution obtained is a numerical or approximated solutions.
- The solutions can be measured in two ways; accuracy and precision.
- Accuracy concept contribute to the concept of true percent relative error.
- Precision contribute to the concept of approximate percent relative error.
- Two major errors involve in numerical methods are round-off and truncation errors.
- Taylor's series is used to approximate the value of the function and being used in deriving the numerical methods.
- Sum of the round-off and truncation errors provide the total numerical error.

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