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Ordinary Differential Equations

Chapter 4C: Fourier Series

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Chapter Description

- Expected Outcomes
  - Find the half-range Fourier cosine series and the half-range Fourier sine series

- References
  - Abdullah, S., Nasir, N.M., Jusoh, R., Aziz, L.A. & Yusoff, W.N.S.W., *Ordinary Differential Equations for Engineering Students*. 2016. Universiti Malaysia Pahang.

Content

4.4 Half-range Series

4.5 Effect of Harmonics

4.4 Half – Range Series

- Sometimes a function is given only half of the range (half of the period), and we are required to find the corresponding (full range) Fourier sine series or Fourier cosine series.
  - **Half-range Fourier sine series**
  - **Half-range Fourier cosine series**
- If $f(t)$ be a given half-range function and $f^*(t)$ is the corresponding periodic function, then the Fourier series of is $f^*(t)$ given by

$$f^*(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right) \qquad \text{(i)}$$

Where the Fourier coefficients are

$$a_0 = \frac{2}{T}\int_{d}^{d+T} f^*(t)dt \qquad \text{(ii)}$$

$$a_n = \frac{2}{T}\int_{d}^{d+T} f^*(t)\cos n\omega t dt \quad n = 1, 2, 3, \ldots \qquad \text{(iii)}$$

$$b_n = \frac{2}{T}\int_{d}^{d+T} f^*(t)\sin n\omega t dt \quad n = 1, 2, 3, \ldots \qquad \text{(iv)}$$

Half-range Fourier sine series

- Is a series that has only sine terms but no cosine terms, thus;

Half-range Fourier sine series → $a_n = 0, \quad n = 1, 2, 3, \ldots$

→ $$\int_{d}^{d+T} f^*(t)\cos n\omega t \;\; dt = 0$$

→ $f^*(t)\cos n\omega t$ is odd

→ $f^*(t)$ is odd

- Therefore the half-range Fourier sine series is given by

$$f^*(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$$

Half-range Fourier cosine series

- Is a series that has only cosine terms but no sine terms, thus;

Half-range Fourier cosine series ⟶ $b_n = 0, \quad n = 1, 2, 3, \ldots$

⟶ $\int_{d}^{d+T} f^*(t) \sin n\omega t \; dt = 0$

⟶ $f^*(t) \sin n\omega t$ is odd

⟶ $f^*(t)$ is even

- Therefore the half-range Fourier cosine series is given by

$$f^*(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t$$

Example 7: Half-range Fourier sine series

Consider the function

$$f(t) = t, \quad 0 < t < 1$$

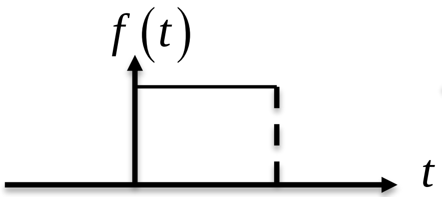
For a half-range Fourier sine series,

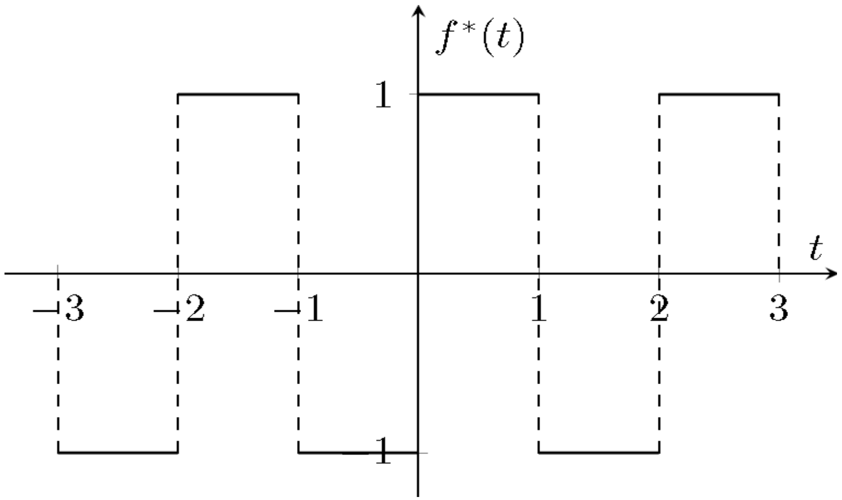
(i) sketch the graph of $f(t)$ and the waveform $f^*(t)$
(ii) write down the analytical description for $f^*(t)$
(iii) show that $b_n = 2(1 - \cos n\pi), \quad n = 1, 2, 3, \ldots$

Obtain the half-range Fourier sine series

Solution

(i)





For a half-range sine series the corresponding periodic function, $f^*(t)$ is odd. Thus the graph of $f^*(t)$ is symmetric about the origin.

(ii) The analytical description of the periodic function $f^*(t)$ is

$$f^*(t) = \begin{cases} -1, & -1 < t < 0 \\ 1, & 0 < t < 1 \end{cases}$$

$$f^*(t) = f^*(t+2)$$

(iii) The period, T = 2 and $\omega = 2$. The nonzero Fourier coefficients,

$$b_n = \frac{2}{2}\int_{-1}^{1} f^*(t)\sin n\pi t dt \qquad \left(f^*(t)\sin n\pi t \text{ is even}\right)$$

$$= 2\times\int_{0}^{1}\sin n\pi t dt$$

$$= -2\left[\frac{\cos n\pi t}{n\pi}\right]_0^1 = \frac{2}{n\pi}(1-\cos n\pi)$$

Now,

$$b_1 = \frac{2}{1\pi}(1-\cos\pi) = \frac{4}{1\pi} \qquad b_2 = \frac{2}{2\pi}(1-\cos 2\pi) = 0$$

$$b_3 = \frac{2}{3\pi}(1-\cos 3\pi) = \frac{4}{3\pi} \qquad b_4 = \frac{2}{4\pi}(1-\cos 4\pi) = 0$$

$$\vdots$$

Therefore, the half-range Fourier sine series is

$$\begin{aligned} f^*(t) &= b_1 \sin \pi t + b_2 \sin 2\pi t + b_3 \sin 3\pi t + b_4 \sin 4\pi t + b_5 \sin 5\pi t + \dots \\ &= \frac{4}{\pi}\sin \pi t + 0 + \frac{4}{3\pi}\sin 3\pi t + 0 + \frac{4}{5\pi}\sin 5\pi t + \dots \\ &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi t}{2n-1} \end{aligned}$$

Example 8: Half-range Fourier cosine series

Consider the function

$$f(t) = 2t, \quad 0 < t < \pi$$

For a half-range Fourier cosine series,

(i) sketch the graph of $f(t)$ and the waveform $f^*(t)$

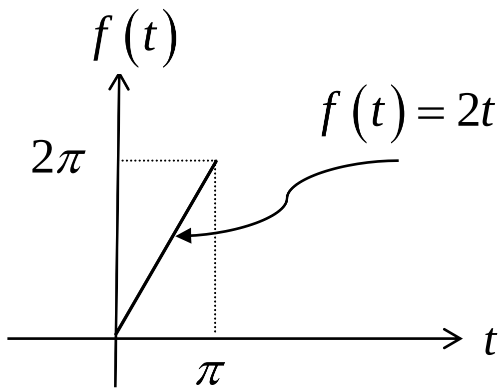
(ii) write down the analytical description for $f^*(t)$

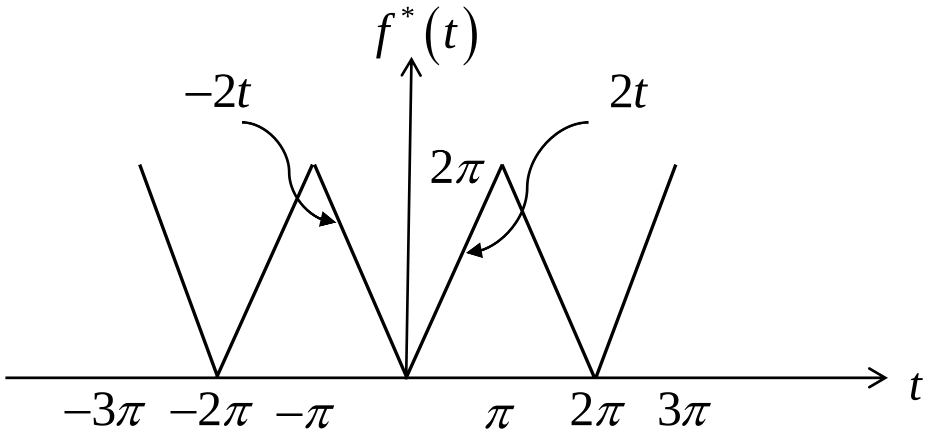
(iii) show that $a_0 = 2\pi$ and $b_n = \dfrac{4}{n^2\pi}(\cos n\pi - 1), \quad n = 1, 2, 3, \ldots.$

Obtain the half-range Fourier cosine series

Solution

(i)





For a half-range cosine series the corresponding periodic function, $f^*(t)$ is even. Thus the graph of $f^*(t)$ is symmetric about the vertical axis.

(ii) The analytical description of the periodic function $f^*(t)$ is

$$f^*(t) = \begin{cases} -2t, & -\pi < t < 0 \\ 2t, & 0 < t < \pi \end{cases}$$

$$= 2|t|, \qquad -\pi < t < \pi$$

$$f^*(t) = f^*(t + 2\pi)$$

(iii) The period, $T = 2\pi$ and $\omega = 1$. Then,

$$a_0 = \frac{2}{2\pi} \int_{-\pi}^{\pi} 2|t| \; dt$$

$$= \frac{2}{\pi} \int_{-\pi}^{\pi} |t| \; dt = \frac{4}{\pi} \int_{0}^{\pi} t \; dt = 2\pi \qquad (|t| \text{ even})$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} 2|t|\cos nt \; dt \qquad (n = 1, 2, 3, ...)$$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi} |t|\cos nt \; dt = 2\times\frac{2}{\pi}\int_{0}^{\pi} t\cos nt \; dt \qquad (|t|\cos nt \;\; \text{even})$$

$$= \frac{4}{\pi}\int_{0}^{\pi} t d\left(\frac{\sin nt}{n}\right) = \frac{4}{\pi}\left\{\left[\frac{t\sin nt}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nt}{n}dt\right\}$$

$$= \frac{4}{\pi}\left\{0 - \left[-\frac{\cos nt}{n^2}\right]_0^{\pi}\right\} = \frac{4}{n^2\pi}(\cos n\pi - 1)$$

Now,

$$a_1 = \frac{4}{1^2\pi}(\cos\pi - 1) = -\frac{8}{1^2\pi} \qquad a_2 = \frac{4}{2^2\pi}(\cos 2\pi - 1) = 0$$

$$a_3 = \frac{4}{3^2\pi}(\cos 3\pi - 1) = -\frac{8}{3^2\pi} \qquad a_4 = \frac{4}{4^2\pi}(\cos 4\pi - 1) = 0$$

$$a_5 = \frac{4}{5^2\pi}(\cos 5\pi - 1) = -\frac{8}{5^2\pi} \qquad a_6 = \frac{4}{6^2\pi}(\cos 6\pi - 1) = 0$$

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The half-range Fourier cosine series is

$$f^*(t) = \pi + a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t + a_4 \cos 4t + a_5 \cos 5t + \dots$$

$$= \pi - \frac{8}{\pi}\left[\cos t + 0 + \frac{1}{9}\cos 3t + 0 + \frac{1}{25}\cos 5t + \dots\right]$$

$$= \pi - \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)t}{(2n-1)^2}$$

Summary of half – range series

Half-range Fourier sine series	$f^*(t)$ odd	The waveform is symmetric about the origin.	$f^*(t)=\sum_{n=1}^{\infty} b_n \sin n\omega t$
Half-range Fourier cosine series	$f^*(t)$ even	The waveform is symmetric about the vertical axis.	$f^*(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n \cos n\omega t$

4.5 Effect of Harmonics

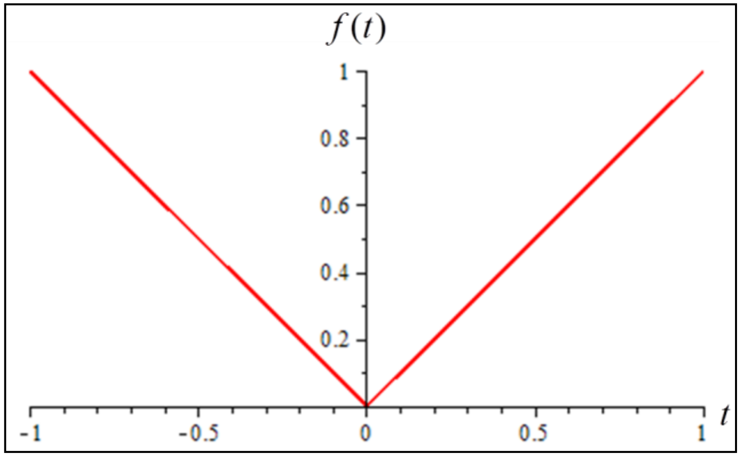
- It is interesting to see how accurate the Fourier series represents the function with which it is associated.
- The complete representation requires an infinite number of terms, but we can at least see the effect of including the first few terms of the series.
- See example 6 for the illustration.

Example 9

The Fourier series of the periodic function

$$f(t) = |t|, \quad -1 \le t < 1$$

$$f(t) = f(t+2)$$

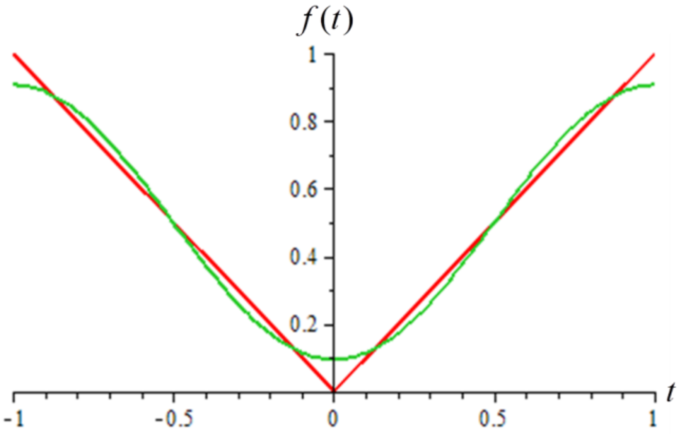


is

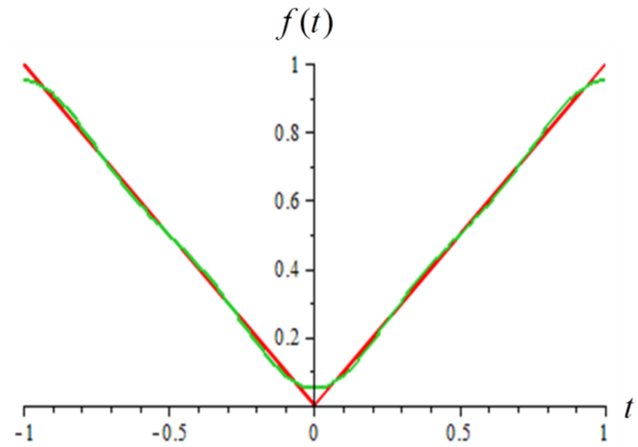
$$f(t) = \frac{1}{2} - \frac{4}{\pi^2}\left(\frac{\cos \pi t}{1} + \frac{\cos 3\pi t}{3} + \frac{\cos 5\pi t}{5} + \ldots\right).$$

If we start with just one cosine term, we can see the effect of including subsequent harmonics. Detailed plotting of points gives the following development.

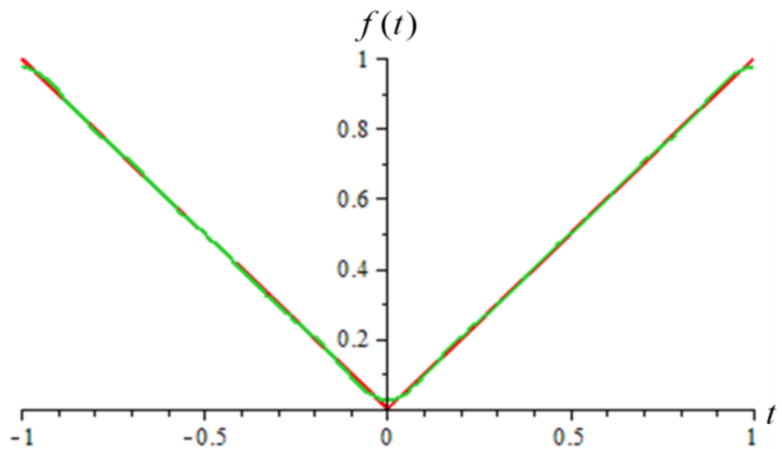
$$f(t)=\frac{1}{2}-\frac{4\cos\pi t}{\pi^2}$$



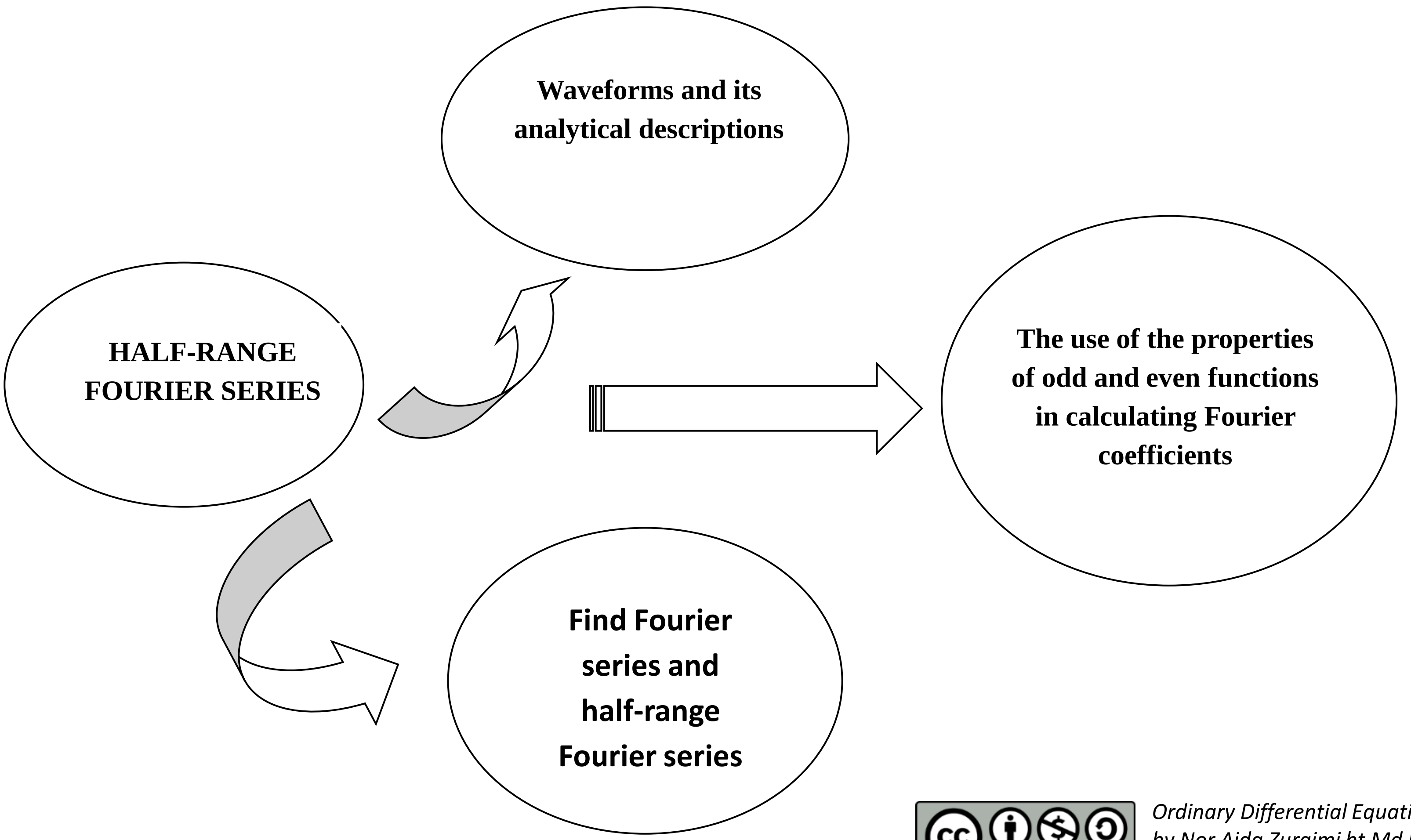
$$f(t)=\frac{1}{2}-\frac{4}{\pi^2}\left(\frac{\cos\pi t}{1}+\frac{\cos 3\pi t}{3}\right)$$



$$f(t)=\frac{1}{2}-\frac{4}{\pi^2}\left(\frac{\cos\pi t}{1}+\frac{\cos 3\pi t}{3}+\frac{\cos 5\pi t}{5}+\frac{\cos 7\pi t}{7}\right)$$



Overview of Fourier Series



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