Ordinary Differential Equations

Chapter 4B: Fourier Series

by
Nor Aida Zuraimi binti Md Noar, Wan Nur Syahidah Wan Yusoff,
Zulkhibri Ismail@Mustofa, Samsudin Abdullah, Nadirah Mohd Nasir,
Rahimah Jusoh@Awang, Laila Amera Aziz

Faculty of Industrial Sciences & Technology

Chapter Description

- Expected Outcomes
  - Find the Fourier series of some periodic functions

- References
  - Abdullah, S., Nasir, N.M., Jusoh, R., Aziz, L.A. & Yusoff, W.N.S.W., *Ordinary Differential Equations for Engineering Students*. 2016. Universiti Malaysia Pahang.

Content

4.3 Full Range Fourier Series

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If $f(t)$ is a periodic function with period *T*, the Fourier series of $f(t)$ is given by

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right) \qquad (4.1)$$

where the Fourier coefficients are

$$a_0 = \frac{2}{T}\int_{d}^{d+T} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_{d}^{d+T} f(t)\cos n\omega t \;dt \qquad n = 1, 2, 3, \ldots$$

$$b_n = \frac{2}{T}\int_{d}^{d+T} f(t)\sin n\omega t \;dt \qquad n = 1, 2, 3, \ldots$$

The choice of *d* is arbitrary as long as the integration is over a period of *T*

Example 4

A periodic function $f(t)$ is given by

$$f(t)=\begin{cases}-2, & -\pi \le t < 0 \\ 2, & 0 \le t < \pi\end{cases}$$

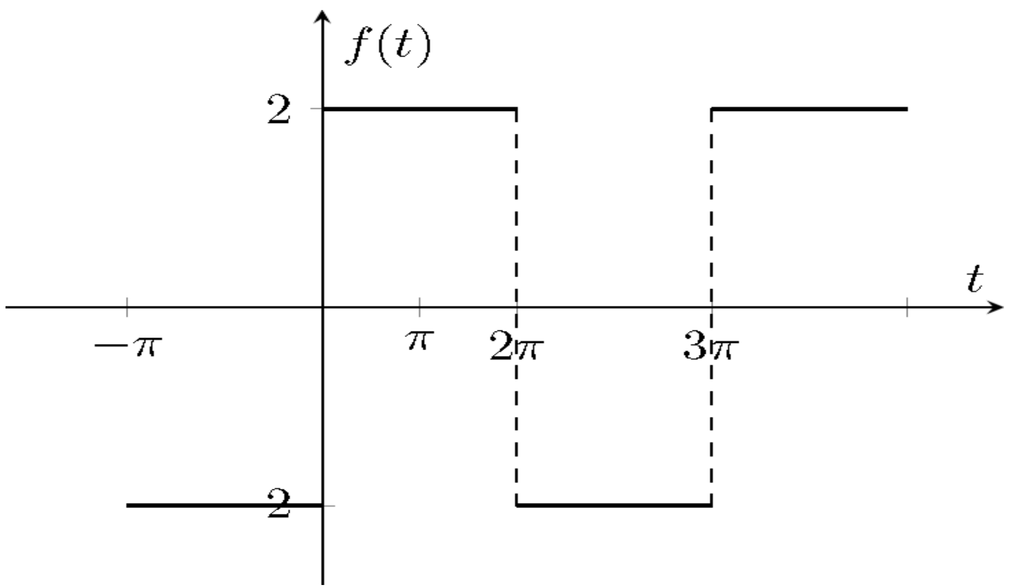
$$f(t)=f(t+2\pi)$$

a) Sketch the waveform of this periodic function over $[-\pi, 3\pi]$

b) Find the Fourier series of $f(t)$

Solution

a)



Since the waveform is symmetric about the origin, $f(t)$ is odd.

Solution

b) Since $T = 2\pi$, we have $\omega = \dfrac{2\pi}{2\pi} = 1$. Now we calculate Fourier coefficients.

$f(t)$ and $f(t)\cos nt$ are odd, then:

$$a_0 = \frac{1}{2}\int_{-\pi}^{\pi} f(t)\ dt = 0$$

$$a_n = \frac{1}{2}\int_{-\pi}^{\pi} f(t)\cos nt\ dt = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\ dt$$

$$= \frac{1}{\pi}\left[2\int_{0}^{\pi} 2\sin nt\ dt\right]$$

$$= \frac{4}{\pi}\left[-\frac{\cos nt}{n}\right]_0^{\pi}$$

$$= \frac{4}{n\pi}\left[1-\cos n\pi\right]$$

Solution

Now,

$$b_1 = \frac{4}{1\pi}\left[1-\cos\pi\right] = \frac{8}{1\pi} \qquad b_2 = \frac{4}{2\pi}\left[1-\cos 2\pi\right] = 0$$

$$b_3 = \frac{4}{3\pi}\left[1-\cos 3\pi\right] = \frac{8}{3\pi} \qquad b_4 = \frac{4}{4\pi}\left[1-\cos 4\pi\right] = 0$$

$$b_5 = \frac{4}{5\pi}\left[1-\cos 5\pi\right] = \frac{8}{5\pi} \qquad b_6 = \frac{4}{6\pi}\left[1-\cos 6\pi\right] = 0$$

$$\vdots \qquad\qquad \vdots$$

Solution

Substituting the above results into Equation (4.1)

$$
\begin{aligned}
f(t) &= \sum_{n=1}^{\infty} b_n \sin nt \\
&= b_1 \sin 1t + b_2 \sin 2t + b_3 \sin 3t + b_4 \sin 4t + b_5 \sin 5t + \ldots \\
&= \frac{8}{\pi}\sin t + 0 + \frac{8}{3\pi}\sin 3t + 0 + \frac{8}{5\pi}\sin 5t + \ldots \\
&= \frac{8}{\pi}\left(\frac{\sin t}{1} + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \ldots\right)
\end{aligned}
$$

$$
f(t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{(2n-1)}
$$

Example 5

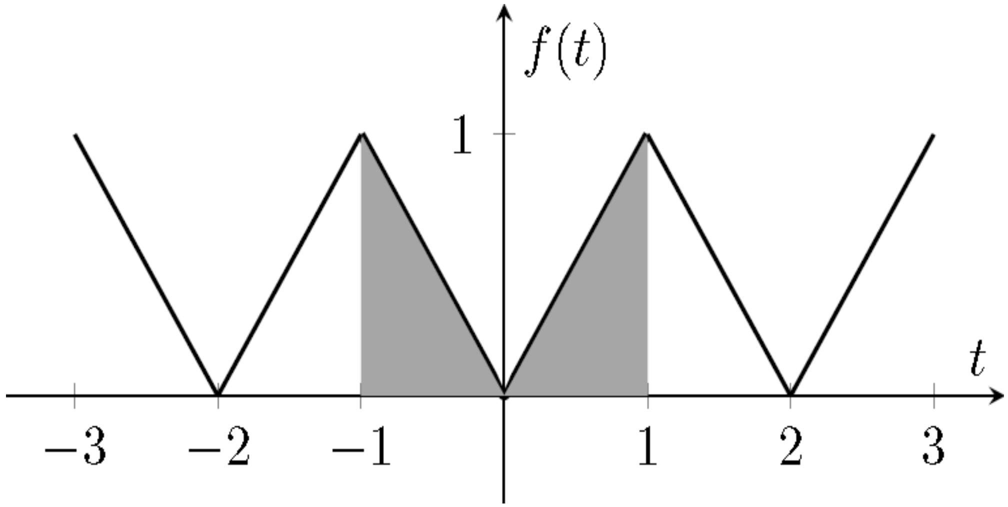
A periodic function $f(t)$ is given by

$$f(t) = |t|, \quad -1 \le t < 1$$
$$f(t) = f(t+2)$$

a) Sketch the waveform of this periodic function over $[-3, 3]$
b) Find the Fourier series of $f(t)$

Solution

a)



Since the waveform is symmetric about the vertical axis, so $f(t)$ is even.

Solution

b) Since $T=2$, then $\omega=\pi$. $f(t)$ is even, hence, $f(t)\sin n\pi t$ is odd.

Therefore,

$$b_n=\frac{2}{2}\int_{-1}^{1} f(t)\sin n\pi t dt=0$$

$$a_0=\frac{2}{2}\int_{-1}^{1} f(t)\ dt=1$$

(the area of the shaded region)

$$a_n=\frac{2}{2}\int_{-1}^{1} f(t)\cos n\pi t\ dt=2\int_{0}^{1} t\cos n\pi t\ dt$$

$$=2\int_{0}^{1} td\left(\frac{\sin n\pi t}{n\pi}\right)=\frac{2}{n\pi}\int_{0}^{1} td(\sin n\pi t)$$

$$=\frac{2}{n\pi}\left[t\sin n\pi t\Big|_0^1-\int_{0}^{1}\sin n\pi t\ dt\right]=\frac{2}{n\pi}\frac{\cos n\pi t}{n\pi}\Bigg|_0^1=\frac{2}{n^2\pi^2}(\cos n\pi-1)$$

Solution

It follows that

$$a_1 = \frac{2}{1^2\pi^2}(\cos\pi - 1) = -\frac{4}{1^2\pi^2}, \qquad a_2 = \frac{2}{2^2\pi^2}(\cos 2\pi - 1) = 0$$

$$a_3 = \frac{2}{3^2\pi^2}(\cos 3\pi - 1) = -\frac{4}{3^2\pi^2}, \qquad a_4 = \frac{2}{4^2\pi^2}(\cos 4\pi - 1) = 0$$

$$a_5 = -\frac{4}{5^2\pi^2} \qquad a_6 = \frac{2}{6^2\pi^2}(\cos 6\pi - 1) = 0$$

Solution

Substituting the above results into Equation (4.1)

$$f(t)=\frac{1}{2}+\sum_{n=1}^{\infty}a_n\cos n\pi t$$

$$f(t)=\frac{1}{2}+\left[-\frac{4}{\pi^2}\cos\pi t-\frac{4}{3^2\pi^2}\cos 3\pi t-\frac{4}{5^2\pi^2}\cos 5\pi t+\ldots\right]$$

$$=\frac{1}{2}-\frac{4}{\pi^2}\left[\frac{\cos\pi t}{1^2}+\frac{\cos 3\pi t}{3^2}+\frac{\cos 5\pi t}{5^2}+\ldots\right]$$

$$=\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi t}{(2n-1)^2}$$

Example 6

A periodic function $f(t)$ is defined analytically by

$$f(t)=\begin{cases}1, & -1<t<0\\ 2, & 0<t<1\end{cases}$$

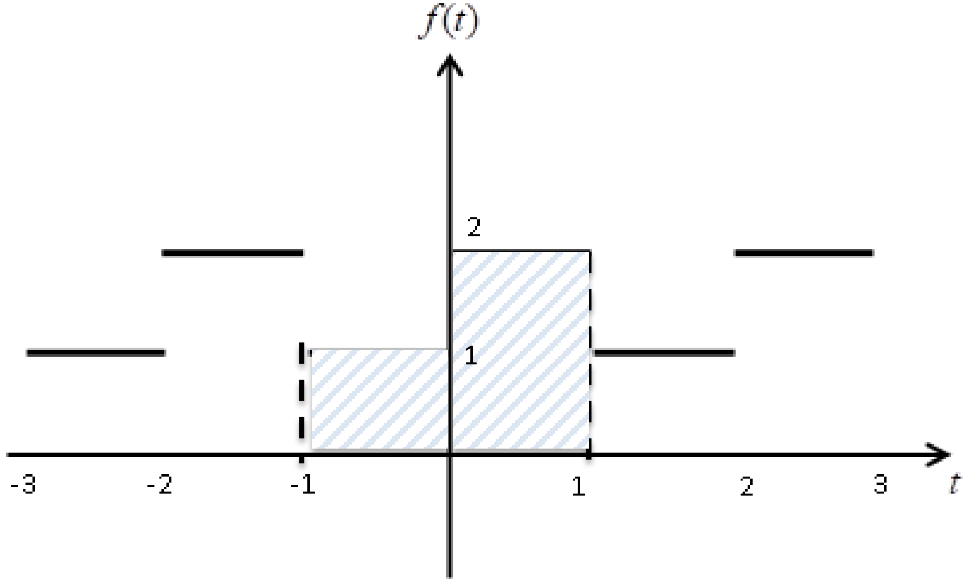
$$f(t)=f(t+2)$$

a) Sketch the waveform of this periodic function over $(-3,3)$

b) Find the Fourier series of $f(t)$

Solution

a)



From the waveform, we find that the graph is not symmetric about the vertical axis as well as the origin. So, $f(t)$ is neither odd nor even.

Solution

b) Since $T = 2$, then $\omega = \pi$. Now we calculate all the Fourier coefficients.

$$a_0 = \frac{2}{2}\int_{-1}^{1} f(t)\ dt = 3 \qquad \text{(the area of the shaded region)}$$

$$a_n = \frac{2}{2}\int_{-1}^{1} f(t)\cos n\pi t\ dt$$

$$= \int_{-1}^{0} \cos n\pi t\ dt + \int_{0}^{1} 2\cos n\pi t\ dt$$

$$= \left[\frac{\sin n\pi t}{n\pi}\right]_{-1}^{0} + \left[\frac{2\sin n\pi t}{n\pi}\right]_{0}^{1}$$

$$= \frac{1}{n\pi}\left[\sin(0) - \sin(-n\pi)\right] + \frac{2}{n\pi}\left[\sin(n\pi) - \sin(0)\right] = 0 \ (\sin n\pi = 0 \text{ for every } n)$$

Solution

$$b_n = \frac{2}{2}\int_{-1}^{1} f(t)\sin n\pi t dt$$

$$= \int_{-1}^{0} \sin n\pi t \; dt + \int_{0}^{1} 2\sin n\pi t \; dt$$

$$= \left[-\frac{\cos n\pi t}{n\pi}\right]_{-1}^{0} + \left[-\frac{2\cos n\pi t}{n\pi}\right]_{0}^{1}$$

$$= -\frac{1}{n\pi}[1-\cos n\pi] - \frac{2}{n\pi}[\cos n\pi - 1] = \frac{1}{n\pi}(1-\cos n\pi)$$

Solution

It follows that

$$b_1 = \frac{1}{1\pi}\left[1-\cos\pi\right] = \frac{2}{1\pi} \qquad b_2 = \frac{1}{2\pi}\left[1-\cos 2\pi\right] = 0$$

$$b_3 = \frac{1}{3\pi}\left[1-\cos 3\pi\right] = \frac{2}{3\pi} \qquad b_4 = \frac{1}{4\pi}\left[1-\cos 4\pi\right] = 0$$

$$b_5 = \frac{1}{5\pi}\left[1-\cos 5\pi\right] = \frac{2}{5\pi} \qquad b_6 = \frac{1}{6\pi}\left[1-\cos 6\pi\right] = 0$$

$$\vdots \qquad\qquad \vdots$$

Solution

The Fourier series is

$$f(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}b_n\sin n\omega t$$

$$=\frac{1}{2}(3)+\left(b_1\sin\pi t+b_2\sin 2\pi t+b_3\sin 3\pi t+b_4\sin 3\pi t+b_5\sin 5\pi t+\ldots\right)$$

$$=\frac{3}{2}+(\frac{2}{1\pi}\sin\pi t+0+\frac{2}{3\pi}\sin 3\pi t+0+\frac{2}{5\pi}\sin 5\pi t+\ldots)$$

$$=\frac{3}{2}+\frac{2}{\pi}(\frac{\sin\pi t}{1}+\frac{\sin 3\pi t}{3}+\frac{\sin 5\pi t}{5}+\ldots)$$

$$f(t)=\frac{3}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi t}{(2n-1)}$$

Summary of Full Range

Fourier series of even function

- Suppose $f(t)$ is an even function of period *T* and defined in the interval $d < t < d + T$ and $f(t)$ be periodic with period *T*.
- Then, the Fourier series can be represented as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t$$

$$a_0 = \frac{2}{T} \int_{d}^{d+T} f(t) dt$$

$$a_n = \frac{2}{T} \int_{d}^{d+T} f(t) \cos n\omega t dt \qquad n = 1, 2, \ldots$$

Summary of Full Range

Fourier series of odd function

- Suppose $f(t)$ is an odd function of period *T* and defined in the interval $d < t < d+T$ and $f(t)$ be periodic with period *T*.
- Then, the Fourier series can be represented as

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$b_n = \frac{2}{T}\int_{d}^{d+T} f(t)\sin n\omega t\,dt \qquad n = 1, 2, \ldots$$

Summary of Full Range

Fourier series of neither even nor odd functions

- If $f(t)$ is neither even nor odd, then Fourier series is of the form

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right)$$

Author Information

Nor Aida Zuraimi binti Md Noar
aidaz@ump.edu.my

Wan Nur Syahidah binti Wan Yusoff
wnsyahidah@ump.edu.my

Zulkhibri Ismail@Mustofa
zulkhibri@ump.edu.my

Nadirah Mohd Nasir
nadirah@ump.edu.my

Rahimah Jusoh@Awang
rahimahj@ump.edu.my

Laila Amera Aziz
laila@ump.edu.my

Samsudin Abdullah
samsudin382@gmail.com