BUM2113 Ordinary Differential Equations

Chapter 3A: Laplace Transforms

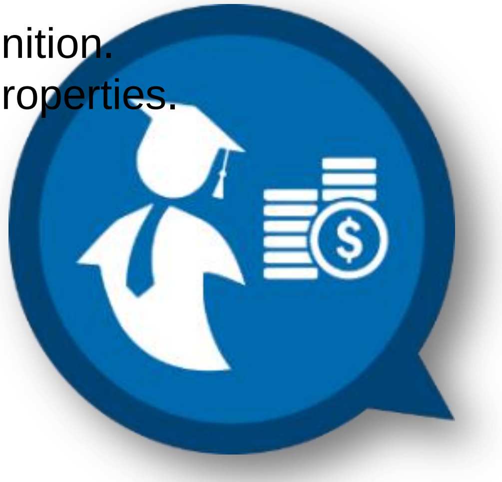
by
Nor Aida Zuraimi binti Md Noar, Laila Amera Aziz,
Wan Nur Syahidah Wan Yusoff, Samsudin Abdullah,
Nadirah Mohd Nasir, Rahimah Jusoh@Awang

Faculty of Industrial Sciences & Technology

Chapter Description

Expected Outcomes

1. Explain the concept of Laplace Transforms.
2. Determine the Laplace Transforms of a function via definition.
3. Determine the Laplace Transforms using its table and properties.



References

Samsudin Abdullah, Nadirah Nasir, Rahimah Jusoh @ Awang, Laila Amera Aziz, Wan Nur Syahidah Wan Yusoff,, Module : Ordinary Differential Equations (BUM2133), 4rd Edition 2016.

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3.0 Introduction

3.1 Definition and Notation

3.2 Properties of the Laplace Transforms

3.1 Definition and Notation

INTRODUCTION

- The attraction of the Laplace transform is that it transforms *differential equations* in the t (time) domain into algebraic equations in the s (frequency) domain.
- Advantage of using the Laplace transform for solving differential equations is that initial conditions play an essential role in the transformation process, so they are automatically incorporated into the solution.
- It is also a very powerful method that is able to solve any order of linear differential equations.

The Laplace transform transforms *differential equations* in the *t* (time) domain into algebraic equations in the *s* (frequency) domain.

$$f(t) \xrightarrow{\mathcal{L}\{\ \}} F(s)$$

Laplace transform operator

Ex: $\mathcal{L}\{f(t)\} = F(s)$

$\mathcal{L}\{g(t)\} = G(s)$

$\mathcal{L}\{y(t)\} = Y(s)$

In general, the function to be transformed is denoted by a lowercase letter, while its Laplace transform will be denoted by the corresponding upper case letter. The Laplace transform is a one-to-one function.

Definition and notation

Let $f(t)$ be a function defined over $[0,\infty)$. Then

$$\boxed{\mathcal{L}\{f(t)\} = \lim_{T\to+\infty}\int_0^T e^{-st} f(t)\,dt = F(s), \qquad s>0}$$

is called the **Laplace Transforms** of if the integral exists.

Laplace Transform via Definition

Example 1:

Use the definition to find the Laplace transform of $f(t)=3$.

$$\mathcal{L}\{3\} = \lim_{T\to+\infty}\int_0^T 3e^{-st}\,dt = 3\lim_{T\to+\infty}\int_0^T e^{-st}\,dt = 3\lim_{T\to+\infty}\left[\frac{e^{-st}}{-s}\right]_0^T = \frac{3}{s}$$

Example 2:

Find the Laplace transform of $f(t)=e^{2t}$.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(a-s)t}dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}, s>a$$

However, the Laplace transform of the

Example 3: Find the Laplace transform of $f(t) = \cos 5t$.

$$\mathcal{L}\{\cos 5t\} = \int_0^\infty e^{-st}\cos 5t\,dt$$

$$= \left[\frac{1}{5}e^{-st}\sin 5t - \frac{s}{25}e^{-st}\cos 5t\right]_0^\infty - \frac{s^2}{25}\int_0^\infty e^{-st}\cos 5t\,dt$$

$$\left(1+\frac{s^2}{25}\right)\int_0^\infty e^{-st}\cos at\,dt = (0-0) - \left(0 - \frac{s}{25}\right)$$

$$\int_0^\infty e^{-st}\cos at\,dt = \frac{s}{25}\left(\frac{25}{25+s^2}\right)$$

$$\therefore \quad \mathcal{L}\{\cos at\} = \frac{s}{s^2+25}$$

Differentiate	Integrate
$(+)\ e^{-st}$	$\cos at$
$(-)\ -se^{-st}$	$\frac{1}{a}\sin at$
$(+)\ s^2e^{-st}$	$-\frac{1}{a^2}\cos at$

However, the generalised form of Laplace transform for certain functions are as summarised in the Laplace table on the next slide.

Laplace Table

$f(t)$	$F(s),\ s>0$
a	$\dfrac{a}{s}$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$
$\sin at$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$

$f(t)$	$F(s),\ s>0$
$\sinh at$	$\dfrac{a}{s^2-a^2}$
$\cosh at$	$\dfrac{s}{s^2-a^2}$
$u(t-d)$	$\dfrac{e^{-sd}}{s}$
$\delta(t-d)$	e^{-sd}
$\delta(t)$	1

Example: Use the Laplace table to determine the Laplace transforms of

(a) $f(t) = t^4$

(b) $f(t) = \sin 6t$

(c) $f(t) = \cosh \frac{2}{3}t$

(d) $f(t) = \delta(t+2)$

Solution:

(a)

$$n = 4$$

$$\mathcal{L}\left\{t^4\right\} = \frac{4!}{s^{4+1}} = \frac{24}{s^5}$$

(b) $a = 6$

$$\mathcal{L}\left\{\sin 6t\right\} = \frac{6}{s^2 + 6^2} = \frac{6}{s^2 + 36}$$

(c)

$$a = \frac{2}{3}$$

$$\mathcal{L}\left\{\cosh \frac{2}{3}t\right\} = \frac{s}{s^2 - \left(\frac{2}{3}\right)^2} = \frac{9s}{9s^2 - 4}$$

(d)

$$d = -2$$

$$\mathcal{L}\left\{\delta(t+2)\right\} = e^{-s(-2)} = e^{2s}$$

3.2 Properties of the Laplace Transforms

3.2.1 LINEARITY PROPERTY

If $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ exists, and α, β are constants, then

$$\boxed{\begin{aligned}\mathcal{L}\{\alpha f(t)+\beta g(t)\} &= \alpha\mathcal{L}\{f(t)\}+\beta\mathcal{L}\{g(t)\}\\ &= \alpha F(s)+\beta G(s)\end{aligned}}$$

Example: Find the Laplace transforms of

(a) $f(t) = 8 - 2\delta(t) + 5\cos 3t$

(b) $f(t) = u(t-\pi) + 7e^{2+3t}$

Solution:

$$\text{(a)}\quad \mathcal{L}\{8 - 2\delta(t) + 5\cos 3t\} = \mathcal{L}\{8\} - 2\mathcal{L}\{\delta(t)\} + 5\mathcal{L}\{\cos 3t\}$$

$$= \frac{8}{s} - 2 + \frac{5s}{s^2+9}$$

$$\text{(b)}\quad \mathcal{L}\{u(t-\pi) + 7e^{2+3t}\} = \mathcal{L}\{u(t-\pi)\} - 7e^2\mathcal{L}\{e^{3t}\}$$

$$= \frac{e^{-\pi s}}{s} - \frac{7e^2}{s-3}$$

3.2.2 FIRST SHIFT PROPERTY

If $\mathcal{L}\{f(t)\} = F(s)$ exists, and *a* is a constant, then

$$\boxed{\mathcal{L}\left\{e^{-at} f(t)\right\} = F(s+a)}$$

$$\mathcal{L}\left\{ e^{-at} \;\vdots\; f(t) \right\} = F(s+a)$$

$$\downarrow \qquad\qquad \downarrow$$

$$a = ? \qquad f(t) \xrightarrow{\mathcal{L}\{\ \}} F(s) \longrightarrow F(s+a)$$

Example : Use the first shift theorem to find the Laplace Transform of

(a) $e^{-t}\sin t$

(b) $e^{2+t}+t^2e^{4t}$

Solution :

(a)

$$a=1, \qquad f(t)=\sin t$$

$$F(s)=\frac{1}{s^2+1}$$

$$F(s+1)=\frac{1}{(s+1)^2+1}$$

$$\therefore \mathcal{L}\left\{e^{-t}\sin t\right\}=\frac{1}{(s+1)^2+1}$$

(b) $\mathcal{L}\left\{e^{2+t}+t^2e^{4t}\right\}=e^2\mathcal{L}\left\{e^t\right\}+\mathcal{L}\left\{t^2e^{4t}\right\}$ by linearity theorem

From the Laplace table,

$$e^2\mathcal{L}\left\{e^t\right\}=\frac{e^2}{s-1}$$

Using the first shift theorem,

$a=-4, \quad f(t)=t^2$

$$F(s)=\frac{2}{s^3}$$

$$F(s-4)=\frac{2}{(s-4)^3}$$

$$\mathcal{L}\left\{t^2e^{4t}\right\}=\frac{2}{(s-4)^3}$$

$$\therefore \mathcal{L}\left\{e^{2+t}+t^2e^{4t}\right\}=\frac{e^2}{s-1}+\frac{2}{(s-4)^3}$$

3.2.3 SECOND SHIFT PROPERTY

If $\mathcal{L}\{f(t)\} = F(s)$ exists, and *a* is a constant, then

$$\boxed{\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)}$$

$$\mathcal{L}\{u(t-a)\,f(t-a)\} = e^{-as}F(s)$$

$$a = ?$$

$$f(t-a) \longrightarrow f(t) \xrightarrow{\mathcal{L}\{\ \}} F(s)$$

Example: Find the Laplace transform of

(a) $\mathcal{L}\left\{u(t-8)(t-8)^2\right\}$

(b) $\mathcal{L}\left\{u(t+1)(7+\delta(t))\right\}$

(c) $\mathcal{L}\left\{(t-2)u(5t^2+t-1)\right\}$

Solution:

(a)

$$a=8, \quad f(t-8)=(t-8)^2$$

$$f(t)=t^2$$

$$F(s)=\mathcal{L}\left\{t^2\right\}=\frac{2}{s^3}$$

$$\therefore \mathcal{L}\left\{u(t-8)(t-8)^2\right\}=e^{-8s}\frac{2}{s^3}$$

(b) $$a=-1,\quad f(t+1)=7+\delta(t)$$

$$=7+\delta\left[(t+1)-1\right]$$

$$\text{*Substitute } t=(t+1)-1$$

$$f(t)=7+\delta(t-1)$$

$$F(s)=\mathcal{L}\left\{7+\delta(t-1)\right\}$$
$$=\frac{7}{s}+e^{-s}$$

$$\therefore \mathcal{L}\left\{u(t+1)\delta(t)\right\}=e^{s}\left(\frac{7}{s}+e^{-s}\right)$$

(c) $a = 2, \quad f(t-2) = 5\boxed{t}^2 + \boxed{t} - 1$

$$= 5\left[(t-2)+2\right]^2 + \left[(t-2)+2\right] - 1$$

$*$Substitute $t = (t-2) + 2$

$$f(t) = 5[t+2]^2 + [t+2] - 1$$
$$= 5t^2 + 21t + 21$$

$$F(s) = \mathcal{L}\left\{5t^2+21t+21\right\}$$
$$= \frac{10}{s^3} + \frac{21}{s^2} + \frac{21}{s}$$

$$\therefore \mathcal{L}\left\{(t-2)u\left(5t^2+t-1\right)\right\} = e^{-2s}\left(\frac{10}{s^3} + \frac{21}{s^2} + \frac{21}{s}\right)$$

3.2.4 DERIVATIVE OF t-TRANSFORM

If $\mathcal{L}\{f(t)\} = F(s)$ exists, and n=1,2,3,…

$$\boxed{\mathcal{L}\{t^n\ f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}}$$

$$\mathcal{L}\{t^n\ f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$$

$$n = ?$$

$$f(t) \xrightarrow{\mathcal{L}\{\ \}} F(s)$$

Example : Use the derivative of t-transform to find the Laplace transform of

(a) $\mathcal{L}\left\{t^2 e^{-t}\right\}$

(b) $\mathcal{L}\left\{te^{-t}u(t-1)\right\}$

Solution :

(a)

$$n=2, \quad f(t)=e^{-t}$$

$$F(s)=\mathcal{L}\left\{e^{-t}\right\}=\frac{1}{s+1}$$

$$\mathcal{L}\left\{t^2 e^{-t}\right\}=(-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s+1}\right)$$

$$=\frac{d}{ds}\left[\frac{d}{ds}(s+1)^{-1}\right]=\frac{d}{ds}\left[-(s+1)^{-2}\right]$$

$$=2(s+1)^{-3}$$

(b)

$$n = 1, \quad f(t) = e^{-t}u(t-1)$$

$$F(s) = \mathcal{L}\left\{e^{-t}u(t-1)\right\} = \frac{e^{-(s+1)}}{s+1}$$

$$\mathcal{L}\left\{te^{-t}u(t-1)\right\} = (-1)^1 \frac{d}{ds}\left(\frac{e^{-(s+1)}}{s+1}\right)$$

$$= -\left[\frac{-e^{-(s+1)}(s+1) - (1)e^{-(s+1)}}{(s+1)^2}\right]$$

by quotient rule

$$= e^{-(s+1)}\left[\frac{s+2}{(s+1)^2}\right]$$

Using first shift theorem:

$$a = 1, \qquad f(t) = u(t-1)$$

$$F(s) = \frac{e^{-s}}{s}$$

$$F(s+1) = \frac{e^{-(s+1)}}{s+1}$$

$$\mathcal{L}\left\{e^{-t}u(t-1)\right\} = \frac{e^{-(s+1)}}{s+1}$$

Author Information

Nor Aida Zuraimi binti Md Noar
aidaz@ump.edu.my

Laila Amera Aziz
laila@ump.edu.my

Nadirah Mohd Nasir
nadirah@ump.edu.my

Rahimah Jusoh@Awang
rahimahj@ump.edu.my

Wan Nur Syahidah binti Wan Yusoff
wnsyahidah@ump.edu.my

Samsudin Abdullah
samsudin382@gmail.com