Ordinary Differential Equations

Chapter 2B: Second Order Differential Equations

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Chapter Description

Expected Outcomes

1. Solve non homogenous equations using the method of variation of parameters methods.

2. Determine and find the general solutions of Euler equations.

References

Samsudin Abdullah, Nadirah Nasir, Rahimah Jusoh @ Awang, Laila Amera Aziz, Wan Nur Syahidah Wan Yusoff,, Module : Ordinary Differential Equations (BUM2133), 4rd Edition 2016.

Content

2.2 Linear Non-Homogenous Equations

2.2.2 The Methods of Variation of Parameters

2.3 Euler Equations

2.4 Applications on Second Order Differential Equations

2.2 Linear non-homogenous constant-coefficient equations

Theorem:

If y_c is a complementary solution of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

and y_p is a particular integral solution of nonhomogeneous equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

then the general solution of nonhomogeneous equation is given by

$$y(x)=y_c+y_p\ .$$

2.2.2 LINEAR NONHOMOGENEOUS CONSTANT-COEFFICIENT EQUATIONS: (METHOD OF VARIATION OF PARAMETERS)

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Recall back:

The previous method can be used to find the solution where the f(x) term has the form of:

(i) $f(x)$ is a polynomial of nth degree

(ii) $f(x) = Ce^{\lambda x}$ where C and λ are constants

(iii) $f(x) = P\cos ax$ or $Q\sin ax$ where P, Q and a are constants

(iv) linear combination or product of the above case.

So now, we will concentrate on solving nonhomogeneous d.e when f(x) has the form of

$$\tan x \quad \cot x \quad \sec x \quad \operatorname{cosec} x \quad \frac{1}{x^n} \quad \text{and} \quad \ln x$$

We will study a technique called variation of parameters which was introduced by a French mathematician, Joseph Louis Langrange(1736-1813M).

In this method, we let the general solution of the nonhomogenous d.e

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

be

$$y=uy_1+vy_2$$

where u and v are functions of x to be determined, and y_1 and y_2 are the solutions of the corresponding homogenous d.e

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

General solution : $y = uy_1 + vy_2$

Below explain the formula to determine u and v.

$$y = uy_1 + vy_2$$

$$u = -\int \frac{y_2 f(x)}{aW} dx$$

$$v = \int \frac{y_1 f(x)}{aW} dx$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. W is known as Wronskian.

S1: Express the given equation in the form of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

S2: Determine the values of ***a*** and ***f(x)***.

S3: Find the complementary solution, $\mathbf{y_c}$.

S4: Calculate the Wronskian, $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

S5: Calculate $u=-\int\frac{y_2 f(x)}{aW}dx+A$ and $v=\int\frac{y_1 f(x)}{aW}dx+A$

S6: Obtain the solution for $y=uy_1+vy_2$

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Example 1:

Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2x^2 - 1$

Solution:

1. $a = 1$ and $f(x) = 2x^2 - 1$

2. Find y_c :

$$m^2 - 2m + 1 = 0$$

$$\therefore \quad m_1 = m_2 = 1$$

$$y_c = (A + Bx)e^x = Ae^x + Bxe^x$$

Thus, $y_1 = e^x$ and $y_2 = xe^x$

$$y_1' = e^x \qquad y_2' = xe^x + e^x$$

3. Calculate the W

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x(x+1) \end{vmatrix} = e^{2x}(x+1) - xe^{2x} = e^{2x}$$

4. Now, we obtain

$$u = -\int \frac{y_2 f(x)}{aW} dx$$

$$= -\int \frac{e^x\left(2x^3 - x\right)}{e^{2x}} dx$$

$$= -\int e^{-x}\left(2x^3 - x\right) dx$$

$$= e^{-x}\left(2x^3 + 6x^2 + 11x + 11\right) + A$$

$$v=\int \frac{y_1 f(x)}{aW}dx$$

$$=\int e^{-x}\left(2x^2-1\right)dx$$

$$=-e^{-x}\left(2x^2+4x+3\right)+B$$

5. Solve for general solution

$$y=uy_1+vy_2$$

$$=\left[e^{-x}\left(2x^3+6x^2+11x+11\right)+A\right]e^x+\left[-e^{-x}\left(2x^2+4x+3\right)+B\right]xe^x$$

$$=\left(A+Bx\right)e^x+2x^2+8x+11$$

Example 2:

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + y = \tan x$$

Solution:

$$1.\ a = 1 \quad \text{and} \quad f(x) = \tan x$$

$$2.\ \text{Find } y_c:$$

$$m^2 + 1 = 0$$

$$\therefore m = \pm i$$

$$y_c = A\cos x + B\sin x$$

$$\text{Thus, } y_1 = \cos x \quad \text{and} \quad y_2 = \sin x$$

$$y_1' = -\sin x \qquad y_2' = \cos x$$

3. Calculate the W

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}=\cos^2 x+\sin^2 x=1$$

4. Now, we obtain

$$u=-\int \frac{y_2 f(x)}{aW}dx$$

$$=-\int \sin x \tan x dx$$

$$=-\int \frac{\sin^2 x}{\cos x}dx$$

$$=-\int \frac{1-\cos^2 x}{\cos x}dx$$

$$=-\int \sec x dx+\int \cos x dx$$

$$=-\ln\left(\sec x+\tan x\right)+\sin x+A$$

$$v = \int \frac{y_1 f(x)}{aW} dx$$

$$= \int \cos x \tan x dx$$

$$= \int \sin x dx$$

$$= -\cos x + B$$

5. Solve for general solution

$$y = uy_1 + vy_2$$

$$= \left[-\ln\left(\sec x + \tan x\right) + \sin x + A \right] \cos x + \left[-\cos x + B \right] \sin x$$

$$= A\cos x + B\sin x - \cos x \ln\left(\sec x + \tan x\right)$$

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Example 3:

Find the general solution of the differential equation $y''-5y'+6=x$

Solution:

1. $a=1$ and $f(x)=x$

2. Find y_c :

$$m^2-5m+6=0$$

$$m_1=3,\ m_2=2$$

$$y_c=A\,e^{3x}+B\,e^{2x}$$

Thus, $y_1=e^{3x}$ and $y_2=e^{2x}$

$$y_1'=3e^{3x} \qquad y_2'=2e^{2x}$$

3. Calculate the W

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & e^{2x} \\ 3e^{3x} & 2e^{2x} \end{vmatrix} = 2e^{5x} - 3e^{5x} = -e^{5x}$$

4. Now, we obtain

$$u = -\int \frac{y_2 f(x)}{aW} dx$$

$$= -\int \frac{(e^{2x})(x)}{(1)(-e^{5x})} dx$$

$$= \int xe^{-3x} dx$$

$$= \frac{xe^{-3x}}{-3} - \left(\frac{e^{-3x}}{(-3)(-3)} \right) = \frac{xe^{-3x}}{-3} - \frac{e^{-3x}}{9} + A$$

$$v=\int \frac{y_1 f(x)}{aW}dx$$

$$=\int \frac{(e^{3x})(x)}{-e^{5x}}dx$$

$$=-\int xe^{3x-5x}dx$$

$$==\frac{xe^{-2x}}{2}+\frac{e^{-2x}}{4}+B$$

5. Solve for general solution

$$y=uy_1+vy_2$$

$$=\left(-\frac{xe^{-3x}}{3}-\frac{e^{-3x}}{9}+A\right)\left(e^{3x}\right)+\left(\frac{xe^{-2x}}{2}+\frac{e^{-2x}}{4}+B\right)\left(e^{2x}\right)$$

$$=-\frac{x}{3}-\frac{1}{9}+Ae^{3x}+\frac{x}{2}+\frac{1}{4}+Be^{2x}$$

$$=Ae^{3x}+Be^{2x}+\frac{x}{6}+\frac{5}{36}$$

2.3 Euler Equations

Euler equation is a special type of second order linear equations where the coefficients are functions of the independent variables. By using a suitable substitution Euler equations can be reduced to equations with constant coefficient.

The linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 x \frac{dy}{dx} + a_0 y = f(x) ---(3)$$

where the coefficient $a_0, a_1, \ldots, a_n$ are constant.

EULER EQUATION OF SECOND ORDER:

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = f(x) ------(4)$$

where the coefficient $a, b,$ and c are constant

By using the substitutions

$$x = e^t, x\frac{dy}{dx} = \frac{dy}{dt} \text{ and } x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

into Equation(3), now this equation reduced to

$$\mathrm{a}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + b\left(\frac{dy}{dt}\right) + cy = f(e^t)$$

or

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = f(e^t),$$

S1: Ascertain that the given d.e is an Euler equation and express in standard form $ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=f(x).$ Hence, determine a,b,c and f(x).

S2: Substitute $x=e^t, x\frac{dy}{dx}=\frac{dy}{dt}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}-\frac{dy}{dt}$ to express the the d.e as $a\frac{d^2y}{dt^2}+(b-a)\frac{dy}{dt}+cy=f(e^t),$

S3: Then find the general solution by using either the method U.C or V.P The solution obtained in term of ***t.***

S4: Substitute back $t=\ln x$ to the general solution to obtain the solution in term of x.

S5: Obtain the particular solution if given the initial condition.

Example (1): Solve the given differential equation $\frac{d^2y}{dx^2}-\frac{2}{x^2}y=0$

Solution:

Multiply the given equation with x^2 so that it become Euler equation

$$x^2\frac{d^2y}{dx^2}-2y=0,$$

where $a=1, b=0, c=-2$ and $f(x)=0$.

Now, substitute $x=e^t, x\frac{dy}{dx}=\frac{dy}{dt}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}-\frac{dy}{dt}$

into eqn and express in the form

$$a\frac{d^2y}{dt^2}+(b-a)\frac{dy}{dt}+cy=f(e^t).$$

We have

$$\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y=0.$$

Now, we have homogenous equation with constant coefficient.

$$y'' - y' - 2y = 0$$

$$m^2 - m - 2 = 0$$

$$m_1 = 2 \text{ and } m_2 = -1$$

$$\therefore y(t) = Ae^{2t} + Be^{-t}$$

$$= A\left(e^t\right)^2 + B\left(e^t\right)^{-1}$$

Substitute back $t = \ln x$ or $x = e^t$, we obtain

$$y(x) = Ax^2 + \frac{B}{x}.$$

Example (2): Solve the given differential equation $x^2\frac{d^2y}{dx^2}+5x\frac{dy}{dx}+3y=\ln x$

Solution:

This is an Euler equation where $a=1, b=5, c=3$ and $f(x)=\ln x$.

Now, substitute $x=e^t, x\frac{dy}{dx}=\frac{dy}{dt}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}-\frac{dy}{dt}$

into eqn and express in the form

$$a\frac{d^2y}{dt^2}+(b-a)\frac{dy}{dt}+cy=f(e^t).$$

We have

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}+3y=t.$$ (Nonhomogenous constant coefficient)

Now, use either method of undetermined coeffient or variation of' parameters in order to solve this d.e.

By using method of undetermined coeffient give

$$y_c = Ae^{-t} + Be^{-3t}$$

and

$$y_p = \frac{1}{3}(t-4)$$

So that,

$$y(t) = y_c + y_p = Ae^{-t} + Be^{-3t} + \frac{1}{3}(t-4)$$

$$= A\left(e^t\right)^{-1} + B\left(e^t\right)^{-3} + \frac{1}{3}(t-4)$$

Substitute back $t = \ln x$ or $x = e^t$, we obtain

$$y(x) = \frac{A}{x} + \frac{B}{x^3} + \frac{1}{3}\ln x - \frac{4}{3}.$$

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