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Ordinary Differential Equations

Chapter 2A: Second Order Differential Equations

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Chapter Description

Expected Outcomes

1. Determine the solution of homogenous second order ordinary differential equations with constant coefficient.

2. Solve non homogenous equations using the method of undetermined.

References

Samsudin Abdullah, Nadirah Nasir, Rahimah Jusoh @ Awang, Laila Amera Aziz, Wan Nur Syahidah Wan Yusoff,, Module : Ordinary Differential Equations (BUM2133), 4rd Edition 2016.

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2.2.1 The Method of Undetermined Coefficients

2.0 Introduction

The general linear differential equation of order *n* has the form

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+a_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=f(x) \qquad (1)$$

where $a_0(x), a_1(x), \ldots, a_n(x)$ and $f(x)$ are functions in *x*. If $a_0(x), a_1(x), \ldots, a_n(x)$ are **all constants**, the differential equation is said to have ***constant coefficients***, otherwise it is said to have ***variable coefficients***.

For *n* = *2*,equation (1) becomes linear differential equation of **second order**. If $f(x)=0$ equation (1) is called ***homogeneous*** differential equation.

If $f(x)\neq 0$ equation (1) is called ***nonhomogeneous*** equation.

Example:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 1)y = 0$$

It is a **homogeneous** linear differential equation with variable coefficients

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 5y = e^{-x}\sin x$$

It is a **nonhomogeneous** linear differential equation with **constant coefficients**

Functions $y_1, y_2, ..., y_n$ are said to be *linearly independent* if the equation

$$c_1 y_1 + c_2 y_2 + ... + c_n y_n = 0$$

has only a trivial solution, for which $c_1 = c_2 = ... c_n = 0$. Conversely, the functions are said to be *linearly dependent* if there is at least one of $c_1, c_2, ... c_n$ is not equal to zero.

Examples :

Determine whether the following sets of functions are linearly dependent or linearly independent.

$$a)\ \cos x \ \ and \ \ 2\cos x$$

$$b)\ 4x^3 \ \ and \ \ -2x^3$$

$$c)\ e^t \ \ and \ \ e^{2t}$$

$$d)\ x^3 - 2x + 4 \ and \ -4x^3 + 8x - 16$$

Solutions :

$$a)\; k_1 \cos x + 2k_2 \cos x = 0$$

$$k_1 \cos x = -2k_2 \cos x$$

$$\cos x = \frac{-2k_2}{k_1} \cos x$$

$$k_1 = 1, \qquad k_2 = -2$$

$$\therefore \textit{linearly dependent}$$

Examples :

Verify that each functions of y is the general solution for the corresponding equation on the right hand side:

$$a)\ y = Ae^{2x} + Be^{-2x} \qquad ;\ y'' - 4y = 0$$

$$b)\ y = A\cos 4x + B\sin 4x \qquad ;\ y'' + 16y = 0$$

2.1 Linear homogenous constant-coefficient equations

Consider the second order linear homogeneous equation with constant coefficient below:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0, a\neq 0\textbf{.......(1)}$$

- Let $y=e^{mx}$ is a solution of the given equation with **m** is any constant.
- Then, $\frac{dy}{dx}=me^{mx}$ **and** $\frac{d^2y}{dx^2}=m^2e^{mx}$
- Substitute the value of $y, \frac{dy}{dx}$ **and** $\frac{d^2y}{dx^2}$ into equation (1). Therefore,

we have $(am^2+bm+c)e^{mx}=0$

- But since e^{mx} is never zero and $y = e^{mx}$ is a solution, this mean $(am^2 + bm + c) = 0.....(2)$

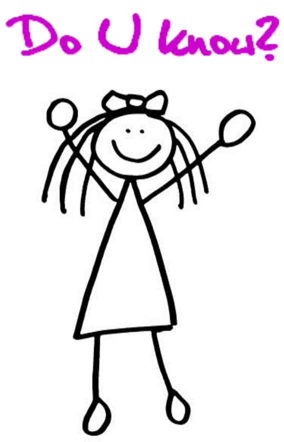
- Equation (2) know as **characteristic or auxiliary equation** for the above DE.

From the discussion, we should notice that from general form;

$$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=0\ ,\ a\neq 0$$

we can find characteristic equation/auxiliary equation

$$am^2+bm+c=0\ ,\ a\neq 0$$



The characteristic equation can be obtained from d.e by replacing:

$$\frac{d^2y}{dx^2}\to m^2$$

$$\frac{dy}{dx}\to m^1$$

$$y\to m^0$$

$$am^2 + bm + c = 0 \ , \ a \neq 0$$ → characteristic equation

Now, the homogenous d.e has apparently been reduced to the purely algebraic problem of finding the roots of c.e. As you know, the **quadratic equation** has three types of roots, if the following conditions are satisfied.

(i) Distinct real roots if $b^2 > 4ac$

(ii) Equal real roots if $b^2 = 4ac$

(iii) Complex/imaginary roots is $b^2 < 4ac$

Therefore, the general solution of homogenous d.e depends on the type of characteristic equation.

Case (i): Distinct real roots $b^2 > 4ac$

→ the c.e will have two different roots m_1 **and** m_2

→ the general solution : $y(x) = Ae^{m_1 x} + Be^{m_2 x}$

Case (ii): Equal real roots $b^2 = 4ac$

→ the c.e will have two equal roots $m_1 = m_2 = m$

→ the general solution : $y(x) = (A + Bx)e^{mx}$

Case (iii): Complex roots $b^2 < 4ac$

→ the c.e will have two complex roots $m_1 = \alpha + \beta i$ **and** $m_2 = \alpha - \beta i$

→ the general solution : $y(x) = e^{\alpha x}(A\cos \beta x + B\sin \beta x)$

Examples : Case(i) Distinct real roots

Find the general solution of $\frac{d^2y}{dx^2}+3\frac{dy}{dx}-40y=0$

Solution:

Characteristic equation is

$$m^2+3m-40=0$$
$$(m-5)(m+8)=0$$
$$m=5 \textbf{ and } m=-8$$

$\therefore$ The general solution is

$$y(x)=Ae^{5x}+Be^{-8x}$$

with A and B is any constant.

Examples : Case(ii) Equal real roots

Find the general solution of $4\frac{d^2y}{dx^2}+12\frac{dy}{dx}+9y=0$

Solution:

Characteristic equation is

$$4m^2+12m+9=0$$

$$(m+\frac{3}{2})(m+\frac{3}{2})=0 \Rightarrow m_1=m_2=-\frac{3}{2}$$

$\therefore$ The general solution is

$$y(x)=Ae^{-\frac{3}{2}x}+Bxe^{-\frac{3}{2}x}=(A+Bx)e^{-\frac{3}{2}x}$$

with A and B is any constant.

Examples : Case(iii) Complex conjugate roots

Find the general solution of $2\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=0$

Solution:

Characteristic equation is

$$2m^2+4m+3=0$$

$$m=-1\pm\frac{1}{\sqrt{2}}i$$

$$\textbf{where } \alpha=-1 \textbf{ and } \beta=\frac{1}{\sqrt{2}}$$

$\therefore$ The general solution is

$$y(x)=e^{-x}\left(A\cos\frac{1}{\sqrt{2}}x+B\sin\frac{1}{\sqrt{2}}x\right).$$

Examples :

Find the particular solution of the following equation satisfying the given conditions

$$y''-4y'+13y=0, \qquad y(0)=-1,\ y'(0)=2$$

Solution :

Characteristic equation:

$$m^2-4m+13=0 \quad \rightarrow \quad a=1, b=-4, c=13$$

$$\therefore \quad m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-4)\pm\sqrt{16-52}}{2}$$

$$=\frac{4\pm\sqrt{-36}}{2}$$

$$=\frac{4\pm 6i}{2}=2\pm 3i \quad \text{where } \alpha=2, \beta=3$$

Therefore the general solution is

$$y = e^{2x}(A\cos 3x + B\sin 3x) \qquad \text{..........(i)}$$

$$y' = 2e^{2x}(A\cos 3x + B\sin 3x) \ +$$
$$e^{2x}(-3A\sin 3x + 3B\cos 3x) \qquad \text{.......... (ii)}$$

Substituting the condition $y(0) = -1$ into (i) we get

$$A = -1$$

Substituting the condition $y'(0) = 2$ into (ii) we get

$$B = 4/3$$

Hence the **particular solution** is

$$y = e^{2x}\left(\frac{4}{3}\sin 3x - \cos 3x\right)$$

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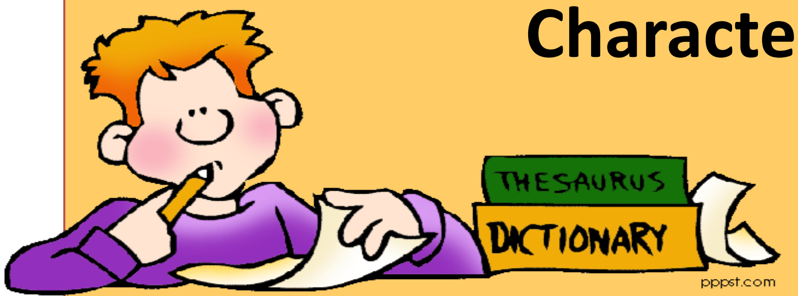
Characteristic Equation :



$$am^2 + bm + c = 0 \ , \ a \neq 0$$

Case	Roots of Characteristic Equation	General Solutions
1	m_1 **and** m_2 real and distinct	$y(x) = Ae^{m_1 x} + Be^{m_2 x}$
2	$m_1 = m_2 = m$ equal real roots	$y(x) = (A + Bx)e^{mx}$
3	$m_1 = \alpha + \beta i$ **and** $m_2 = \alpha - \beta i$ complex roots	$y(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

2.2 Linear non-homogenous constant-coefficient equations

Theorem:

If y_c is a complementary solution of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

and y_p is a particular integral solution of nonhomogeneous equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

then the general solution of nonhomogeneous equation is given by

$$y(x)=y_c+y_p\ .$$

Procedure for finding the **general solution** for nonhomogenous d.e.

For the general solution of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

Find the general solution of, y_c of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

Find the general solution of, y_p of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

$$y=y_c+y_p$$

2.2.1 LINEAR NONHOMOGENEOUS CONSTANT-COEFFICIENT EQUATIONS: (METHOD OF UNDETERMINED COEFFICIENT)

Now, we will study the ***method of undetermined coefficients*** in order to determine particular integrals y_p for linear nonhomogeneous equations

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x) \quad \text{-----(i)}$$

This method is applicable only when the nonhomogenous term *f(x)* on the r.h.s of Equation(i) is one of this term;

CASE (1): $f(x)$ is a polynomial of nth degree

CASE (2): $f(x)=Ce^{\lambda x}$ where C and λ are constants

CASE (3): $f(x)=P\cos ax$ or $Q\sin ax$ where P, Q and a are constants

$f(x)$	Assumed trial solution, $y_p(x)$
$A_0 + A_1 x + \ldots + A_n x^n$	$B_0 + B_1 x + \ldots + B_n x^n$
$e^{\alpha x}$	$ke^{\alpha x}$
$\cos \beta x$ or $\sin \beta x$	$p\cos \beta x + q\sin \beta x$
$\left(A_0 + A_1 x + \ldots + A_n x^n\right)e^{\alpha x}$	$\left(B_0 + B_1 x + \ldots + B_n x^n\right)e^{\alpha x}$
$\left(A_0 + A_1 x + \ldots + A_n x^n\right)\begin{cases}\cos \beta x \\ \sin \beta x\end{cases}$	$\left(B_0 + B_1 x + \ldots + B_n x^n\right)\cos \beta x + \left(C_0 + C_1 x + \ldots + C_n x^n\right)\sin \beta x$
$e^{\alpha x}\begin{cases}\cos \beta x \\ \sin \beta x\end{cases}$	$e^{\alpha x}(p\cos \beta x + q\sin \beta x)$
$\left(A_0 + A_1 x + \ldots + A_n x^n\right)e^{\alpha x}\begin{cases}\cos \beta x \\ \sin \beta x\end{cases}$	$\left(B_0 + B_1 x + \ldots + B_n x^n\right)e^{\alpha x}\cos \beta x + \left(C_0 + C_1 x + \ldots + C_n x^n\right)e^{\alpha x}\sin \beta x$
Sums of any or some of the above entries	Sums of the corresponding trial solutions

CASE 1 : *f*(*x*) is a polynomial of n[th] degree

$f(x)$	y_p
$P_n(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0$	$B_n x^n + B_{n-1}x^{n-1} + \ldots + B_1 x + B_0$

Example

Find the solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2x^2 - 1$$

Solution :

Find y_c : Repeat what you have done for homogeneous d.e.

$$m^2 - 2m + 1 = 0$$

$$(m-1)(m-1) = 0$$

$$\therefore \quad m = 1$$

$$y_c = (A + Bx)e^x$$

Find y_p : We can see that nonhomogeneous term $f(x) = 2x^2 - 1$
So,

$$f(x) = 2x^2 - 1 \longrightarrow y_p = ax^2 + bx + c$$

Then,

$$y_p = ax^2 + bx + c$$

$$y_p' = 2ax + b$$

$$y_p'' = 2a$$

Substitute this result into $y'' - 2y' + y = 2x^2 - 1$

We get

$$2a-2(2ax+b)+ax^2+bx+c=2x^2-1$$
$$ax^2+(-4a+b)x+2a-2b+c=2x^2-1$$

Equating the coefficients of

$$x^2: \quad a=2$$
$$x: \quad -4a+b=0 \qquad \therefore b=8$$
$$x^0: 2a-2b+c=-1 \qquad \therefore c=11$$

So, $y_p = 2x^2+8x+11$

Therefore

$$y = y_c + y_p$$
$$= (A+Bx)e^x + 2x^2 + 8x + 11$$

CASE 2 : $f(x) = Pe^{kx}$

$f(x)$	y_p
Pe^{kx}	Ce^{kx}

Example

Find the solution of the differential equation

$$\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 4e^{5x}$$

Solution :

Find y_c : Repeat what you have done for homogeneous d.e.

$$m^2 + 14m + 49 = 0$$

$$(m+7)(m+7) = 0$$

$$\therefore \quad m = -7 \; (repeated)$$

$$y_c = e^{-7x}(A + Bx)$$

Find y_p : We can see that nonhomogeneous term $f(x) = 4e^{5x}$

So,

$$f(x) = 4e^{5x} \longrightarrow y_p = Ce^{5x}$$

Then,

$$y_p = Ce^{5x}$$

$$y_p' = 5Ce^{5x}$$

$$y_p'' = 25Ce^{5x}$$

Substitute $y_{p,}\ y_p'$ *and* y_p'' *in the given DE*

$$25Ce^{5x} + 14(5Ce^{5x}) + 49Ce^{5x} = 4e^{5x}$$

$$25Ce^{5x} + 70Ce^{5x} + 49Ce^{5x} = 4e^{5x}$$

$$144C = 4 \rightarrow C = \frac{1}{36}$$

Thus,

$$y_p = \frac{1}{36}e^{5x}$$

Therefore

$$y = y_c + y_p$$

$$y = e^{-7x}(A + Bx) + \frac{1}{36}e^{5x}$$

CASE 3 : $f(x) = P\sin kx \quad OR \quad P\cos kx$

$f(x)$	y_p
$P\sin kx$	$C\cos kx + D\sin kx$
$P\cos kx$	$C\cos kx + D\sin kx$
$P\cos kx + Q\sin kx$	$C\cos kx + D\sin kx$
$P\cos kx + Q\sin mx$	$x^r(C\cos kx + D\sin kx)$ $+x^r(E\cos mx + F\sin mx)$

Example

Find the solution of the differential equation

$$y''-5y'+6y=2\sin 4x$$

Solution :

Find y_c : Repeat what you have done for homogeneous d.e.

$$m^2-5m+6=0$$

$$(m-3)(m-2)=0$$

$$m=3, m=2$$

$$y_c=Ae^{3x}+Be^{2x}$$

Find y_p : We can see that nonhomogeneous term $f(x) = 2\sin 4x$

So,

$f(x) = 2\sin 4x \longrightarrow y_p = C\cos 4x + D\sin 4x$

Choose $r = 0$ (smallest) so that no term in y_p which is common with those of y_c

Then, $f(x) = 2\sin 4x \rightarrow y_p = x^r (C\cos 4x + D\sin 4x)$

$$for\ r = 0$$

$$y_p = C\cos 4x + D\sin 4x$$

$$y_p' = -4C\sin 4x + 4D\cos 4x$$

$$y_p'' = -16C\cos 4x - 16D\sin 4x$$

Substitute $y_{p,}\ y_p'$ and y_p'' in the given DE

$$y''-5y'+6y=2\sin 4x$$

$$-16C\cos 4x-16D\sin 4x-5(-4C\sin 4x+4D\cos 4x)+6(C\cos 4x+D\sin 4x)=2\sin 4x$$

$$-16C\cos 4x-16D\sin 4x+20C\sin 4x-20D\cos 4x+6C\cos 4x+6D\sin 4x=2\sin 4x$$

$$(20C-10D)\sin 4x-(10C+20D)\cos 4x=2\sin 4x$$

Equate the coefficient:

$$\sin 4x : 20C-10D=2\,......(1)\times 2 \quad \rightarrow 40C-20D=4....(1)$$

$$\cos 4x : -10C-20D=0...(2) \quad \rightarrow -10C-20D=0...(2)$$

$$50C=4 \quad \rightarrow C=\frac{2}{25}$$

$$D=-\frac{1}{25}$$

Solution : $y=y_c+y_p$

$$y=Ae^{3x}+Be^{2x}+\frac{1}{25}\left(2\cos 4x-\sin 4x\right)$$

CASE 4 : Linear combination (addition or subtraction) of case 1, case 2 and case3

$f(x)$	y_p
$f(x) = x^2 \pm Pe^{kx}$	$(Cx^2 + Dx + E) + (Fe^{kx})$
$f(x) = Pe^{kx} \pm Q\sin kx$	$(Ce^{kx}) + (D\cos kx + E\sin kx)$
: :	: :

Example

Find the solution of the differential equation

$$y'' + 4y = e^x - 2$$

Solution :

Find y_c : Repeat what you have done for homogeneous d.e.

$$m^2 + 4 = 0$$

$$m = \sqrt{-4}$$

$$m = \pm j2$$

$$y_c = A\cos 2x + B\sin 2x$$

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Find y_p : We can see that nonhomogeneous term $f(x)=e^x-2$

So,

$f_1(x)=e^x$ $\longrightarrow$ $y_{p1}=x^r(Ce^x)$

Choose $r=0$ ($\because$ no similar term)

Now, $y_{p1}=Ce^x$

$f_2(x)=-2$ $\longrightarrow$ $y_{p2}=x^rD$

Choose $r=0$ ($\because$ no similar term)

Now, $y_{p2}=D$

$$\text{Then, } y_p = y_{p1}+y_{p2}$$
$$= Ce^x + D$$
$$y_p' = Ce^x$$
$$y_p'' = Ce^x$$

Please continue.....

CASE 5 : Product of case 1, case 2 and case3

$f(x)$	y_p
$P_n(x)e^{\lambda x}$	$x^r(Cx^n + Dx^{n-1} + \ldots + E)e^{\lambda x}$
$P_n(x)\begin{cases}\cos\beta x \\ \sin\beta x\end{cases}$	$x^r(Cx^n + Dx^{n-1} + \ldots + E)\cos\beta x$ $+$ $x^r(Fx^n + Gx^{n-1} + \ldots + I)\sin\beta x$
$Ce^{\lambda x}\begin{cases}\cos\beta x \\ \sin\beta x\end{cases}$	$x^r e^{\lambda x}(C\cos\beta x + D\sin\beta x)$
$P_n(x)Ce^{\lambda x}\begin{cases}\cos\beta x \\ \sin\beta x\end{cases}$	$x^r e^{\lambda x}(Cx^n + Dx^{n-1} + \ldots + E)\cos\beta x$ $+$ $x^r e^{\lambda x}(Fx^n + Gx^{n-1} + \ldots + I)\sin\beta x$

Example

Find the solution of the differential equation

$$y''+4y'+3y=xe^{-x}$$

Solution :

Find y_c : Repeat what you have done for homogeneous d.e.

$$m^2+4m+3=0$$

$$m_1=-3 \;\&\; m_2=-1$$

$$y_c=Ae^{-3x}+Be^{-x}$$

Find y_p : We can see that nonhomogeneous term $f(x) = xe^{-x}$

So,

$f(x) = xe^{-x}$ ⟶ $y_p = x^r (Cx + D)e^{-x}$

Choose $r = 1$ (smallest) so that no term in y_p which is common with those of y_c

Then,

$$f(x) = xe^{-x} \rightarrow y_p = x^r (Cx + D)e^{-x}$$

$$for\ r = 1$$

$$y_p = x(Cx + D)e^{-x} = \left(Cx^2 + Dx\right)e^{-x}$$

$$y_p' = \left(2Cx + D\right)e^{-x} - \left(Cx^2 + Dx\right)e^{-x}$$

$$y_p'' = 2Ce^{-x} - \left(2Cx + D\right)e^{-x} - \left(2Cx + D\right)e^{-x} + \left(Cx^2 + Dx\right)e^{-x}$$

Please continue.....

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