Ordinary Differential Equations

Chapter 1B : First Order Ordinary Differential Equations

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Chapter Description

Expected Outcomes

1. Identify the types of the first-order ordinary differential equations i.e:
   - Homogeneous Equations
   - Linear Equations; and
   - Exact Equations.

3. Find the solution of the first-order ordinary differential equations by using the above mention methods.

4. Model physical problems in the form of the first-order ordinary differential equations and solve them.

References

– Samsudin Abdullah, Nadirah Nasir, Rahimah Jusoh @ Awang, Laila Amera Aziz, Wan Nur Syahidah Wan Yusoff,, Module : Ordinary Differential Equations (BUM2133), 4rd Edition 2016.

Content

1.4 Homogenous Equations

How to identify whether or not a given equation is homogeneous?

The differential equation

$$\frac{dy}{dx} = f(x, y)$$

is called homogeneous equation if

$$f(\lambda x, \lambda y) = f(x, y)$$

for every real value of λ

Example : Determining homogeneous equations

Identify whether the following is a separable differential equation

$$(a)\ \frac{dy}{dx}=\frac{y-x}{y+x} \qquad (b)\ \frac{dy}{dx}=x^2y+y^2$$

Solution:

(a) Given $\frac{dy}{dx}=\frac{y-x}{y+x}$

$$f(x,y)=\frac{y-x}{y+x}$$

$$f(\lambda x,\lambda y)=\frac{\lambda y-\lambda x}{\lambda y+\lambda x}$$

$$=\frac{\lambda(y-x)}{\lambda(y+x)}=f(x,y)$$

Therefore, the equation is homogeneous

(b) Given $\frac{dy}{dx}=x^2y+y^2$

$$f(x,y)=x^2y+y^2$$

$$f(\lambda x,\lambda y)=(\lambda x)^2(\lambda y)+(\lambda y)^2$$

$$=\lambda^3x^2y+\lambda^2y^2$$

Since, we cannot eliminate the λ therefore, the equation is not homogeneous

Solving homogeneous equations :

In practice, homogeneous equations can be reduced to separable equations, using the substitution

$$y = vx \quad \text{or} \quad v = \frac{y}{x} \qquad (1)$$

From that, we have

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad (2)$$

Step summary of solving homogeneous equations

Step 1 : Make sure that the given equation is homogeneous

Step 2 : Substitute equations (1) and (2) into the original equation.

Step 3 : Separate the variables x and v in the resulting equation.

Step 4 : Integrate both sides of the **DE**. w.r.t. the related variables.

Step 5 : Substitute back $v = \frac{y}{x}$ to obtain the general solution.

Example : Solving DE by using homogeneous equations

Solve $\dfrac{dy}{dx} = \dfrac{xy}{x^2 + y^2}$

Solution: This **DE** is a homogeneous equation

Using substitution $y = vx$, we have

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{x(vx)}{x^2 + (vx)^2}$$

$$= \frac{v}{1+v^2}$$

$$x\frac{dv}{dx} = \frac{v}{1+v^2} - v$$

$$= \frac{v - v - v^3}{1+v^2} = \frac{-v^3}{1+v^2}$$

or

$$\left(\frac{1+v^2}{v^3}\right)dv = -\frac{dx}{x}$$

$$\left(\frac{1}{v^3} + \frac{1}{v}\right)dv = -\frac{dx}{x}$$

Integrate both sides of the equation,

$$\int\left(\frac{1}{v^3} + \frac{1}{v}\right)dv = -\int\frac{dx}{x}$$

we get,

$$-\frac{1}{2v^2} + \ln|v| = -\ln|x| + C, \; C \text{ a constant.}$$

$$\ln|xv| = C + \frac{1}{2v^2}$$

$$\ln|y| = C + \frac{x^2}{2y^2}$$

$$y = e^{C + \frac{x^2}{2y^2}} = Ae^{x^2/2y^2}, A = e^C$$

In the previous examples, the differential equations are homogeneous. However, there are also differential equations which are not homogeneous, by appropriate substitutions can be reduced to homogeneous equations.

HOMOGENEOUS EQUATIONS BY SUBSTITUTION

Example : Solving DE using homogeneous equations by substitution

By using a substitution equation $x = X + 1, \ \ y = Y - 1$

Find the general solution of

$$\frac{dy}{dx} = \frac{2x + y - 1}{x + 2y + 1}$$

HOMOGENEOUS EQUATIONS BY SUBSTITUTION

Example : Solving DE using homogeneous equations by substitution

Solution :

$$x = X + 1, \quad y = Y - 1$$

$$\frac{dx}{dX} = 1 \Rightarrow \qquad dx = dX$$

$$\frac{dy}{dY} = 1 \Rightarrow \qquad dy = dY$$

$$\text{Therefore} \quad \frac{dy}{dx} = \frac{dY}{dX}$$

$$\frac{dy}{dx} = \frac{2x + y - 1}{x + 2y + 1}$$

will becomes

$$\frac{dY}{dX} = \frac{2(X+1) + (Y-1) - 1}{(X+1) + 2(Y-1) + 1} = \frac{2X + Y}{X + 2Y}$$

$$Y = VX, \qquad \frac{dY}{dX} = V + X\frac{dV}{dX}$$

$$V + X\frac{dV}{dX} = \frac{2X + VX}{X + 2(VX)} = \frac{2 + V}{1 + 2V}$$

$$X\frac{dV}{dX} = \frac{2 + V}{1 + 2V} - V$$

$$= \frac{2 + V - V - 2V^2}{1 + 2V}$$

$$X\frac{dV}{dX} = \frac{2 - 2V^2}{1 + 2V} = \frac{2(1 - V^2)}{1 + 2V}$$

$$\int \frac{1 + 2V}{1 - V^2}\, dV = \int \frac{2}{X}\, dX$$

Example : Solving DE using homogeneous equations by substitution

Solution : Continue…

$$\int \frac{1+2V}{1-V^2}\,dV = \int \frac{2}{X}\,dX$$

$$\frac{1+2V}{1-V^2} = \frac{1+2V}{(1+V)(1-V)}$$

Resolve to partial fraction

$$= -\frac{1}{2(1+V)} + \frac{3}{2(1-V)}$$

$$\frac{1}{2}\int -\frac{1}{(1+V)} + \frac{3}{(1-V)}\,dV = \int \frac{2}{X}\,dX$$

$$\frac{1}{2}\left[-\ln|1+V| - 3\ln|1-V|\right] = 2\ln X + C$$

$$\ln|1+V| + 3\ln|1-V| + 4\ln X = \ln D$$

$$\ln D = C$$

$$\ln\left|(1+V)(1-V)^3 X^4\right| = \ln D$$

$$(1+V)(1-V)^3 X^4 = D$$

$$\left(1+\frac{Y}{X}\right)\left(1-\frac{Y}{X}\right)^3 X^4 = D$$

$$\left(\frac{X+Y}{X}\right)\left(\frac{X-Y}{X}\right)^3 X^4 = D$$

$$(X+Y)(X-Y)^3 = D$$

$$(x-1+y+1)(x-1-y-1)^3 = D$$

$$(x+y)(x-y-2)^3 = D$$

1.5 Linear Equations

How do we know that an equation is a linear equation?
An equation is said to be linear if it has the form of

$$a(x)\frac{dy}{dx}+b(x)y=f(x)$$

where $a(x)$, $b(x)$ and $f(x)$ are continuous functions of x or constant.

- To solve a linear equation, we need to transform any given equation to a general form of

$$\frac{dy}{dx}+p(x)y=q(x)$$

← The most general form of first-order linear DE

Where $p(x)=\dfrac{b(x)}{a(x)}$ and $q(x)=\dfrac{f(x)}{a(x)}$.

Solving homogeneous equations :

Step 1 : Rearrange the equation to be in the form of

$$\frac{dy}{dx} + p(x)y = q(x)$$

Step 2 : Get $p(x)$ and $q(x)$ and solve $\int p(x)\,dx$. Integrate without adding constant

Step 3 : Calculate the integrating factor $\rho(x) = e^{\int p\,dx}$

Step 4 : Rearrange the equation in the form

$$\frac{d}{dx}(\rho y) = \rho q(x)$$

Step 5 : Integrate equation (Step 4) with respect to x, which is the solution $\rho y = \int \rho q\,dx$

Explanation on Step 4 :

Suppose, we multiply the integrating factor, $\rho = \rho(x)$ to differential equation $\frac{dy}{dx} + py = q$ when $p = p(x)$ and $q = q(x)$.

And we will obtained $\rho \frac{dy}{dx} + \rho p y = \rho q$

$$\frac{d}{dx}(\rho y) = \rho q$$

Example : Solving DE by using linear equations

Find the particular solution of the differential equation

$$x\frac{dy}{dx}+y=x$$

which satisfies the condition $x = 1$ when $y = 0$.

Solution:

Rearrange $\frac{dy}{dx}+\frac{y}{x}=1$

Define $p(x)=\frac{1}{x}$ and $q(x)=1$.

$$\int p(x)\ dx=\int\frac{1}{x}\ dx=\ln x$$

Integrating factor,

$$\rho=\rho(x)=e^{\int p(x)dx}=e^{\ln x}=x$$

$$\frac{d}{dx}(\rho y)=\rho q,\ \frac{d}{dx}(xy)=x(1)$$

$$d(xy)=\int x\,dx$$

$$xy=\frac{x^2}{2}+C,\qquad y=\frac{x}{2}+\frac{C}{x}$$

$$y(1)=0,\qquad 0=\frac{1}{2}+C,\qquad C=-\frac{1}{2}$$

$$y=\frac{x}{2}-\frac{1}{2x}$$

In the previous examples, the differential equations are linear. However, there are some **nonlinear differential equations that can be reduced to linear equation** by using appropriate substitutions.

LINEAR EQUATIONS BY SUBSTITUTION

Example : Solving DE using linear equations by substitution

By using a substitution equation $z = y^{-2}$

Find the general solution of

$$\frac{dy}{dx} + y = xy^3$$

LINEAR EQUATIONS BY SUBSTITUTION

Example : Solving DE using linear equations by substitution

Solution :

$$z = y^{-2} \qquad (1)$$

$$\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$-\frac{1}{2}\frac{dz}{dx} = y^{-3}\frac{dy}{dx} \qquad (2)$$

Rearrange **DE**,

$$\frac{dy}{dx} + y = xy^{3},$$

$$y^{-3}\frac{dy}{dx} + y^{-2} = x \qquad (3)$$

Substitute
(1) and (2)
(2) into (3)

$$-\frac{1}{2}\frac{dz}{dx} + z = x$$

Solve by using linear equation

$$\frac{dz}{dx} - 2z = -2x$$

$$p = -2 \text{ and } q = -2x$$

$$\rho = e^{\int -2dx} = e^{-2x}$$

$$e^{-2x}z = \int -2xe^{-2x}\,dx$$

$$= xe^{-2x} + \frac{e^{-2x}}{2} + C$$

$$z = x + \frac{1}{2} + Ce^{2x}$$

$$y^{-2} = x + \frac{1}{2} + Ce^{2x}$$

1.6 Exact Equations

How do we know that **DE** is an exact differential equation?

Equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be an exact equation if there are exist a continuous function $f(x,y) = C, C$ constant.

Therefore, the $df = M(x,y)\,dx + N(x,y)\,dy$

If $f = f(x,y)$ then $df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$

Therefore $\dfrac{\partial f}{\partial x} = M(x,y)$ and $\dfrac{\partial f}{\partial y} = N(x,y)$

Since $f(x,y)$ is a continuous function, then

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \quad \text{or} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example : Determining exact equations

Determine whether or not the following differential equations are exact.

$$(a)\quad xy\,dx+\left(\frac{x^2}{2}+\frac{1}{y}\right)dy=0$$

Solution:

$$\frac{\partial u}{\partial x}=xy, \qquad \frac{\partial u}{\partial y}=\frac{x^2}{2}+\frac{1}{y}$$

$$\frac{\partial^2 u}{\partial y\partial x}=x, \qquad \frac{\partial^2 u}{\partial x\partial y}=x$$

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}$$

Therefore this is an **exact** differential equation

$$(b)\quad \left(\frac{x}{y}+3x^2\right)dx+\left(1+\frac{x^3}{y}\right)dy=0$$

$$\frac{\partial u}{\partial x}=\frac{x}{y}+3x^2, \qquad \frac{\partial u}{\partial y}=1+\frac{x^3}{y}$$

$$\frac{\partial^2 u}{\partial y\partial x}=-\frac{x}{y^2}, \qquad \frac{\partial^2 u}{\partial x\partial y}=\frac{3x^2}{y}$$

$$\frac{\partial^2 u}{\partial y\partial x}\neq\frac{\partial^2 u}{\partial x\partial y}$$

Therefore this is **not an exact** differential equation

Solving exact equations :

Step 1 : Rearrange the equation to be in the form of

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Step 2 : If exact, let $\dfrac{\partial f}{\partial x} = M(x,y)\cdots(1)$ and $\dfrac{\partial f}{\partial y} = N(x,y)\cdots(2)$

Or $\dfrac{\partial u}{\partial x} = M\cdots(1)$ and $\dfrac{\partial u}{\partial y} = N\cdots(2)$

Step 3 : Integrate equation (1) w.r.t. x or equation (2) w.r.t. y.

In the case w.r.t. x, we have $f = \int M\,dx + \phi(y)\cdots(3)$

Step 4 : Differentiate Eqn.(3) w.r.t y and equating the result with equation (2) to determine the arbitrary function $\phi(y)$

Step 5 : Write down the solution in the form $f(x,y) = A$, where *A* is a constant.

Example : Solving DE by using exact equations

Solve $\left(6x^2 - 10xy + 3y^2\right)dx + \left(-5x^2 + 6xy - 3y^2\right)dy = 0$

Solution:

Test for exactness:

We know that : $M\left(x, y\right)dx + N\left(x, y\right)dy = 0$

This gives:

$$M\left(x, y\right) = 6x^2 - 10xy + 3y^2 \Rightarrow \frac{\partial M}{\partial y} = -10x + 6y$$

$$N\left(x, y\right) = -5x^2 + 6xy - 3y^2 \Rightarrow \frac{\partial N}{\partial x} = -10x + 6y$$

Therefore,the given equation is an exact differential equation because $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example : Solving DE by using exact equations

Solution : (continue)

Since the equation is an exact equation, we can now find the solution for this **DE**

We know that

$$\frac{\partial f}{\partial x} = M(x, y)$$
$$= 6x^2 - 10xy + 3y^2 \cdots (1)$$
$$\frac{\partial f}{\partial y} = N(x, y)$$
$$= -5x^2 + 6xy - 3y^2 \cdots (2)$$

Integrate (1) with respect to x, we get

$$f = 2x^3 - 5x^2y + 3xy^2 + \phi(y) \cdots (3)$$

where $\phi(y)$ is a function in y.

Differentiate (3) with respect to y, we have

$$\frac{\partial f}{\partial y} = -5x^2 + 6xy + \phi'(y)$$

Equate this equation with equation (2),

$$\phi'(y) = -3y^2$$

Integrate the function, will give us

$$\phi(y) = -y^3 + A, A \text{ constant}$$

The general solution is

Let $f(x, y) = k, k$ constant

$$2x^3 - 5x^2y + 3xy^2 - y^3 + A = k$$

or

$$2x^3 - 5x^2y + 3xy^2 - y^3 = B, \quad B = k - A$$

Modelling with First Order Differential Equations

1. Newton's Cooling Law

$$\frac{d\theta}{dt} = -k\left(\theta - \theta_s\right)$$

where $\theta = \theta(t)$ is the temperature of the cooling object at time t,

θ_s the temperature of the environment,

k is a thermal constant related to the subject,

θ_0 be the initial temperature of the liquid.

2. Electrical Circuits

RL circuit which consists of a resistor, R, an inductor L, and a voltage source, V connected in series.

$$L\frac{di}{dt} + Ri = V$$

where i is the current in amperes and t is the time in seconds.

Modelling with First Order Differential Equations (continue…)

3. Chemical Reaction

$$\frac{dy}{dt} = ky$$

4. Chemical Mixture

$$Q' + \frac{Q}{20} = C$$

5. Population Growth

$$\frac{dP}{dt} = kP + N$$

6. Velocity an airplane

$$m\frac{dv}{dt} = T - bv^2$$

Conclusion

- Differential equations play a fundamental role in real life especially in engineering because many physical phenomena are best formulated mathematically in terms of their rate of change.
- There are five classical methods used in this chapter, which are direct integration, separable equations, homogeneous equations, linear equations and exact equations.
- Some differential equations can be solved using only one method, but some differential equations can use more than one method, as long as the conditions are met.
- Certain differential equations can not be solved directly by using these methods, but by using the substitution equation, the equations becomes solvable.

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