Ordinary Differential Equations

Chapter 1A : First Order Ordinary Differential Equations

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Chapter Description

Expected Outcomes

1. Classify the differential equations according to the order, degree and the linearity of the given differential equations.
2. Identify the types of the first-order ordinary differential equations i.e:
    - Separable Equations

References

– Samsudin Abdullah, Nadirah Nasir, Rahimah Jusoh @ Awang, Laila Amera Aziz, Wan Nur Syahidah Wan Yusoff,, Module : Ordinary Differential Equations (BUM2133), 4rd Edition 2016.

Content

1.0 Introduction

An equation involving derivatives of one or more **dependent variables** with respect to one or more **independent variables**

- Those that are differentiated

- The variables with respect to which differentiation occurs

Ex:

$$\frac{df}{dx}$$

f → Dependent Variables (*DV*)

x → Independent Variables (*IV*)

1.1 Basic Definitions and Terminologies

By Type:

ODE : a differential equation involving ordinary derivatives of one or more dependent variables w.r.t. a single independent variables.

$$6\frac{df}{dx} - 4x\frac{dg}{dx} = \cos x$$

PDE : a differential equation involving partial derivatives of one or more dependent variables w.r.t. more than one independent variables.

$$\frac{\partial^3 f}{\partial x \partial y^2} = 4x^2 + 2y$$

ORDER:

- The highest derivatives in a differential equation
- The order of an equation **is not affected by any power** to which the derivatives may be raised.

Example :

$$\frac{d^2 f}{dx^2} - 4x\frac{df}{dx} = \cos 2x$$

$$\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - 3x = 8$$

DEGREE:

- The degree of a differential equation is the exponent of the highest derivative in the equation

Example :

$$\left(\frac{d^2 f}{dx^2}\right)^2 - 4x\left(\frac{df}{dx}\right)^3 = \cos 2x$$

LINEARITY:

1. Linear differential equation is easier to solve.
2. A differential equation is said to be linear if:
   - the **dependent variable** and it **derivatives** occur to the **first power** only.
   - there are **no products** involving the **dependent variable** with its **derivatives**.
   - there are **no non-linear functions** of the **dependent variable** (e.g: sine, exponent)
3. Otherwise, non-linear.

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 4x^2 + 2y \qquad \frac{\partial^3 f}{\partial x \partial y^2} = 4x^2 + 2y$$

are linear

$$4\frac{dx}{dt} + \sin x = 0 \qquad \left(\frac{dx}{dt}\right)^2 + 4\frac{dx}{dt} = 0$$

are nonlinear

The Solution of a Differential Equation :

There are 2 types of solution for differential equation

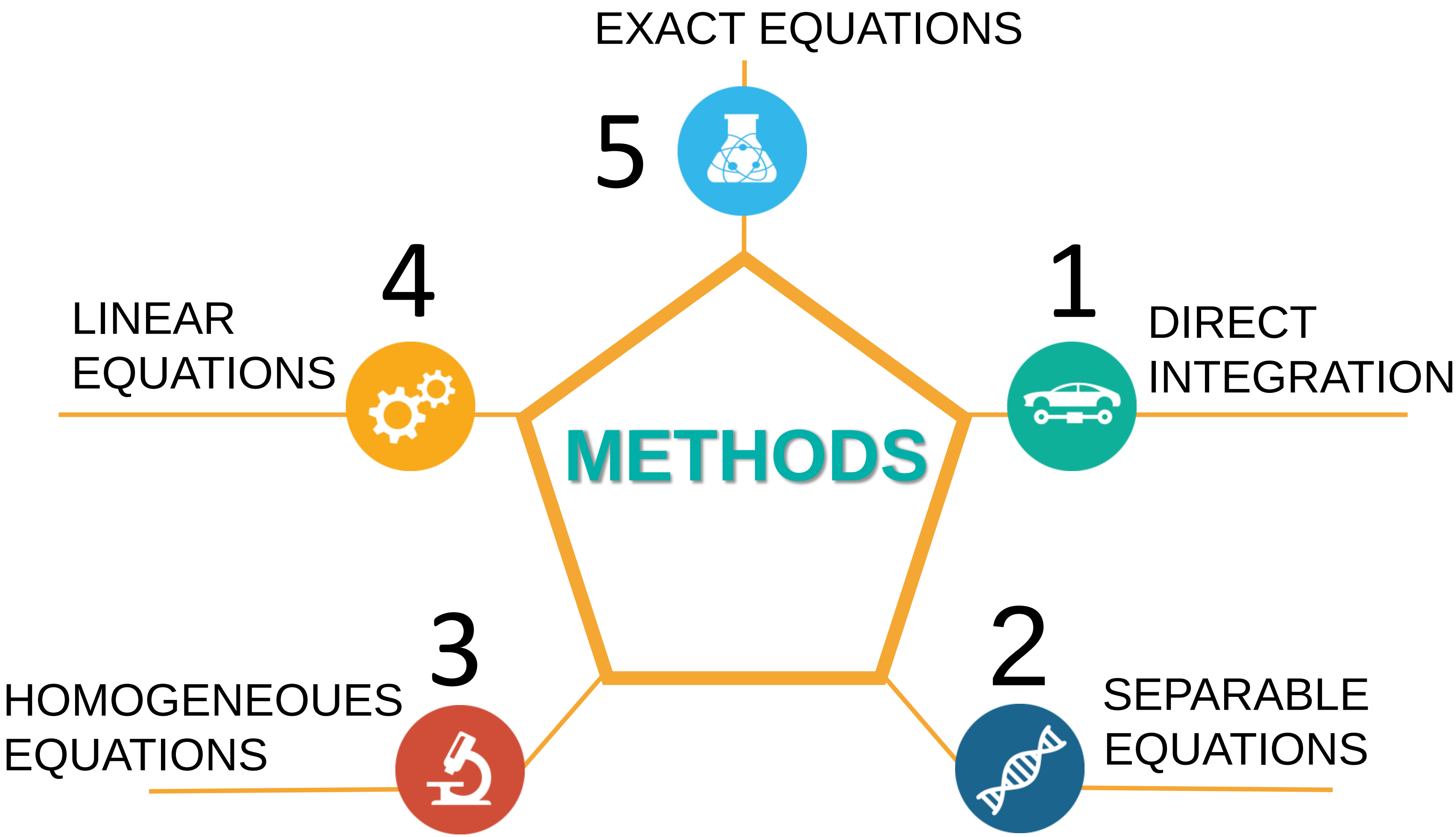
General and Particular Solutions

- The most general function that will satisfy the **DE** contains one or more arbitrary constants is known as the **general solution.**
- The general solution normally contains a number of arbitrary constants equal to the order of the DE.
- Integrate the differential equation : $\frac{dy}{dx} = 2x$

  → On integrating, we obtain a **general solution** $y = x^2 + C$

- Giving particular numerical values to the constant in the general solution result in a **particular solution** of the equation.

OVERVIEW OF FIRST ORDER DIFFERENTIAL EQUATIONS



1.2 Direct Integration

Suppose an ordinary differential equation in the form of

$$\frac{dy}{dx} = f(x).$$

The equation can be solved by integrating both sides with respect to x

$$\int \frac{dy}{dx}\, dx = \int f(x)\, dx$$

$$\int dy = \int f(x)\, dx$$

$$y = \int f(x)\, dx$$

Therefore, technique of **direct integration** can be simplified as follows

$$\text{If } \frac{dy}{dx} = f(x), \text{ then } y = \int f(x)\, dx$$

1.3 Separable Equations

We consider a class of first-order ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

Equation (1) is considered to be **separable** if it can be written in the form

$$h(y)\frac{dy}{dx} = g(x) \tag{2}$$

The solution of equation (2) is obtained by integrating both sides of the equation with respect to x.

$$\int h(y)\frac{dy}{dx}\,dx = \int g(x)\,dx \tag{3}$$

Thus, equation (3) becomes

$$\int h(y)\,dy = \int g(x)\,dx \tag{4}$$

This technique called **separable equations** because we rearrange the equation to be solved such that all terms involving the **dependent variable appear on one side** of the equation, and all terms involving the **independent variable** appear on the other. Integration completes the solution.

$$\frac{dy}{dx} = g(x)h(y)$$

$$\int h(y)\, dy = \int g(x)\, dx$$

Example : Determining separable equations

$$\frac{dy}{dx} = y^2 x e^{3x+4y}$$

$$= (xe^{3x})(y^2 e^{4y})$$

$$= g(x)h(y)$$

separable

$$\frac{dy}{dx} = y + \sin x$$

non separable

Example : Determining separable equations

Identify whether the following is a separable differential equation

$$(a)\ \frac{dy}{dx} = y + 3x^2 y \qquad (b)\ \frac{dy}{dx} = \frac{x+y}{x}$$

Solution:

$$(a)\ \frac{dy}{dx} = y + 3x^2 y$$

can be written as

$$\frac{dy}{dx} = y(1+3x^2)$$

$$\frac{dy}{y} = (1+3x^2)dx$$

Hence, it is a **separable** differential equation.

$$(b)\ \frac{dy}{dx} = \frac{x+y}{x}$$

The given equation is **not a separable** differential equation.

Solving separable equations :

- Step 1: Rewrite the separable equation $\frac{dy}{dx} = g(x)h(y)$ in separated form $\frac{1}{h(y)}dy = g(x)dx$
- Step 2: Integrate each side of this equation w.r.t its respective variable.

Example : Solving DE by using separable equations

Solve $\frac{dy}{dx} = \frac{1+x}{y^2}$

Solution :

Separating variables:

$$y^2\,dy = (1+x)\,dx$$

Integrating

$$\int y^2\,dy = \int (1+x)\,dx$$

$$\frac{y^3}{3} = x + \frac{x^2}{2} + C$$

$$y^3 = 3x + \frac{3x^2}{2} + 3C$$

$$y^3 = 3x + \frac{3x^2}{2} + D, \quad D = 3C$$

Example : Solving DE by using separable equations

Solve the given differential equation $x\frac{dy}{dx}=1+y$

Solution :

We write $x\frac{dy}{dx}=1+y$ as

$$\frac{dy}{1+y}=\frac{dx}{x}$$

Integrate both sides

$$\int\frac{dy}{1+y}=\int\frac{dx}{x}$$

We get

$$\ln|1+y|=\ln|x|+C \quad ,C \text{ constant.}$$

$$\ln|1+y|-\ln|x|=C$$

$$\ln\left|\frac{1+y}{x}\right|=C$$

$$\left|\frac{1+y}{x}\right|=e^C,\ D=e^C, D \text{ constant.}$$

$$\text{or } y=Dx-1.$$

In the previous examples, the differential equations are separable. However, there are also **inseparable differential equations that can be reduced to separable equations** by appropriate substitutions.

SEPARABLE EQUATIONS BY SUBSTITUTION

Example : Solving DE using separable equations by substitution

By using a substitution equation $z = x + y,$

find the general solution of

$$\frac{dy}{dx} = \frac{x+y}{1-x-y}$$

SEPARABLE EQUATIONS BY SUBSTITUTION

Example : Solving DE using separable equations by substitution

By using a substitution equation $z = x + y,$ find the general solution of

$$\frac{dy}{dx} = \frac{x+y}{1-x-y}$$

Solution :

$$\frac{dy}{dx} = \frac{x+y}{1-x-y} \qquad (1)$$

$$z = x + y \qquad (2)$$

$$\frac{dz}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 1 \qquad (3)$$

Substitute (2) and (3) into (1)

$$\frac{dz}{dx} - 1 = \frac{z}{1-z}$$

$$\frac{dz}{dx} = \frac{z}{1-z} + 1 = \frac{z+1-z}{1-z} = \frac{1}{1-z}$$

$$\int (1-z)\,dz = \int 1\,dx$$

$$z - \frac{z^2}{2} = x + C$$

Replace back equation (2) to have answer in term of y as the *DV*

$$x + y - \frac{(x+y)^2}{2} = x + C$$

$$2y - 2xy - y^2 = x^2 + D, \quad D = 2C$$

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