DISCRETE MATHEMATICS AND APPLICATIONS

Proof Techniques 2

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Chapter Description

Chapter Outline

4.4 Mathematical Induction

4.5 Strong Induction and Well-Ordering

Aims

- Apply mathematical induction to prove a theorem
- Apply strong induction to prove a theorem as an alternative for mathematical induction

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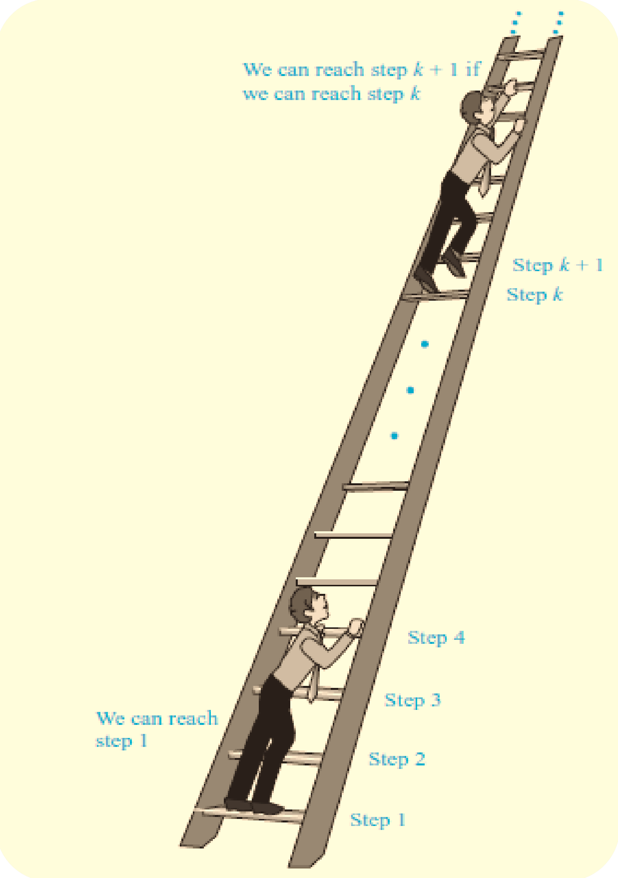
Mathematical Induction: Introduction

Suppose that we have an infinite ladder, and we want to know whether we can reach every step on this ladder.

✓ We can reach the first rung of the ladder. *P*(1) is true

✓ If we can reach a particular rung of the ladder, then we can reach the next rung.
P(*k*) is true → *P*(*k+1*) is true



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Mathematical Induction: Steps

Given a statement $P(n)$. We need to prove that $P(n)$ is always true for the given $n \in \{n_0, n_0+1, n_0+2, \ldots\}$. There are four steps given as follows:

Step 1: Prove that $P(n_0)$ is true.

(n_0 is refer to the first value of n given in the theorem)

Step 2: Assume that $P(k)$ is true for some n.

Step 3: Prove that $P(k+1)$ is true.

Step 4: Conclusion. The statement is true for every integer $n \geq n_0$.

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Mathematical Induction: Example 1 (i)

Let n be natural number and $n \geq 1$. Prove that $1+2+3+...+n = \dfrac{n(n+1)}{2}$.

Proof:

Let $P(n)$ be the predicate $1+2+3+...+n = \dfrac{n(n+1)}{2}$. In this example, $n_0 = 1$.

Step 1: Show that $P(1)$ is true.

LHS: 1 and RHS: $\dfrac{1(1+1)}{2} = 1$

Since LHS=RHS, then $P(1)$ is true.

Step 2: Assume that $P(k)$ is true for some $k \geq 1$.

Assume that $1+2+3+...+k = \dfrac{k(k+1)}{2}$ is true for some $k \geq 1$.

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Mathematical Induction: Example 1 (ii)

Step 3: Show that $P(k+1)$ must also be true.

We now wish to show the truth of $P(k+1)$,

i.e $1+2+3+...+(k+1)=\frac{(k+1)((k+1)+1)}{2}$

$$\begin{aligned} \text{LHS } 1+2+3+...+(k+1) &= 1+2+3+...+k+(k+1) \\ &= \frac{k(k+1)}{2}+(k+1) \qquad \text{by Step 2} \\ &= (k+1)\left(\frac{k}{2}+1\right) \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2} = RHS \end{aligned}$$

Thus we have shown $P(k+1)$ is true.

Step 4: By the principle of mathematical induction, it follows that

$1+2+3+...+n=\frac{n(n+1)}{2}$ is true for all $n \geq 1$.

Mathematical Induction: Example 2 (i)

Show that $2+2^2+2^3+\ldots+2^n=2^{n+1}-2$ for $n\in\mathbb{N}=\{1,2,3,\ldots\}$

Proof:

Let $P(n)$ be the predicate $2+2^2+2^3+\ldots+2^n=2^{n+1}-2$. In this example, $n_0=1$.

Step 1: Show that $P(1)$ is true.

LHS: 2 and RHS: $2^{1+1}-2=2$

Since LHS=RHS, then $P(1)$ is true.

Step 2: Assume that $P(k)$ is true for some $k\geq 1$.

Assume $2+2^2+2^3+\ldots+2^k=2^{k+1}-2$ is true for some $k\geq 1$.

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Mathematical Induction: Example 2 (ii)

Step 3: Show that $P(k+1)$ must also be true.

Need to show that $2+2^2+2^3+...+2^{k+1}=2^{k+1+1}-2$ is true.

$$
\begin{aligned}
LHS \quad 2+2^2+2^3+...+2^{k+1} &= 2+2^2+2^3+...+2^k+2^{k+1} \\
&= 2^{k+1}-2+2^{k+1} && \text{by assumption in Step 2} \\
&= 2\cdot 2^{k+1}-2 \\
&= 2^{k+1+1}-2 = RHS
\end{aligned}
$$

Therefore $P(k+1)$ is true.

Step 4: By the principle of mathematical induction, it follows that $2+2^2+2^3+...+2^n=2^{n+1}-2$ is true for $n\in\mathbb{N}=\{1,2,3,...\}$

Mathematical Induction: Example 3 (i)

Show that $2^n > n$ for all $n = 1, 2, 3, \ldots$.

Proof:

Let $P(n)$ be the predicate $2^n > n$. In this example, $n_0 = 1$.

Step 1: Show that $P(1)$ is true.

$2^1 > 1 \Rightarrow 2 > 1$ is true.

Step 2: Assume that $P(k)$ is true for some integers n.

Assume $2^k > k$ is true for some integers n.

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Mathematical Induction: Example 3 (ii)

Step 3: Show that $P(k+1)$ must also be true.

Need to show that $2^{k+1} > k+1$ is true.

$$2^k > k \quad \text{by assumption in Step 2}$$

$$2^k \cdot 2 > k \cdot 2$$

$$2^{k+1} > 2k = k + k \geq k + 1$$

$$2^{k+1} > k + 1$$

Thus $2^{k+1} > k+1$. Therefore $P(k+1)$ is true.

Step 4: By the principle of mathematical induction, it follows that $2^n > n$ is true for all $n = 1, 2, 3, \ldots$.

Strong Induction & Well ordering

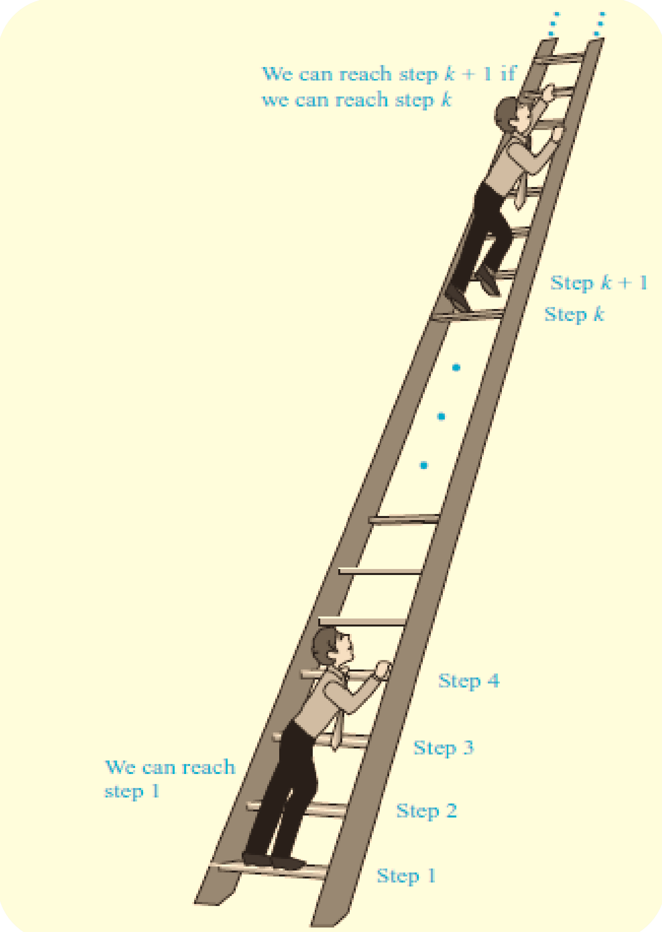
Sometimes mathematical induction cannot use easily, then another form of mathematical induction, called **strong induction** can be used.

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, two steps are needed:

- ✓ Show that the proposition $P(1)$ is true.
- ✓ Show that the conditional statement

  $(P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \rightarrow P(k+1)$ is true

  for all positive integers k.



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Strong Induction: Steps

Here are the steps to prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function.

Step 1: Show that $P(1)$ *is true*

Step 2: Assume that $P(j)$ is true for all integers j with $1 \leq j \leq k$

Step 3: Show that $P(k+1)$ *is true*

Step 4: Conclusion

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Strong Induction: Example 1 (i)

Show that if n is an integer greater than 1,
then n can be written as the product of primes.

Proof:

Let $P(n)$ be the proposition that n can be written as the product of primes.

Step 1: Show that P(1) is true

$P(1)$ is true, because 2 can be written as the product of one prime, itself.

***Step 2: Assume* that $P(j)$ is true for all integers j with $2 \le j \le k$**

Assume that j can be written as the product of primes whenever j is a positive integer at least 2 and not exceeding k

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Strong Induction: Example 1 (ii)

Step 3: Show that P(k+1) is true

There are two cases to consider, namely, when $k + 1$ is prime and when $k + 1$ is composite.

Case I: $k + 1$ is prime.

If $k + 1$ is prime, we immediately see that $P(k + 1)$ is true

Case II: $k + 1$ is composite.

$k + 1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < k + 1$ where both a and b are integers at least 2 and not exceeding k, we can use the inductive hypothesis to write both a and b as the product of primes. Thus, if $k + 1$ is composite, it can be written as the product of primes.

Step 4: Conclusion

If n is an integer greater than 1, then n can be written as the product of primes.

Strong Induction: Example 2 (i)

Consider a game in which two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game. Show that if the two piles contain the same number of matches initially, the second player can always guarantee a win.

https://www.youtube.com/watch?v=Hof7P__P68I

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Strong Induction: Example 2 (ii)

Proof:

<u>Step 1: Show that P(1) is true</u>

When $n = 1$, the first player has only one choice, removing one match from one of the piles, leaving a single pile with a single match, which the second player can remove to win the game.

<u>Step 2: Assume that P(j) is true for all integers j with 2 ≤ j ≤ k</u>

Assume that the second player can always win whenever there are j matches, where $1 \le j \le k$ in each of the two piles at the start of the game.

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Strong Induction: Example 2 (ii)

Step 3: Show that P(k+1) is true

Need to show that $P(k + 1)$ is true, that is, that the second player can win when there are initially $k + 1$ matches in each pile.

Suppose that there are $k + 1$ matches in each of the two piles at the start of the game and suppose that the first player removes r matches $(1 \leq r \leq k)$ from one of the piles, leaving $k + 1 - r$ matches in this pile. By removing the same number of matches from the other pile, the second player creates the situation where there are two piles each with $k + 1 - r$ matches. Because $1 \leq k + 1 - r \leq k$, we can now use the inductive hypothesis to conclude that the second player can always win.

Step 4: Conclusion

If the first player removes all $k + 1$ matches from one of the piles, the second player can win by removing all the remaining matches is true

Well Ordering

The validity of both the principle of mathematical induction and strong induction follows from a fundamental axiom of the set of integers, the **well-ordering property.**

Definition: Well-ordering property.

The well-ordering property states that every nonempty set of nonnegative integers has a least element.

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Axioms for the set of positive integers

The axioms we now list specify the set of positive integers as the subset of the set of integers satisfying four key properties. We assume the truth of these axioms in this textbook.

- **Axiom 1** The number 1 is a positive integer.
- **Axiom 2** If n is a positive integer, then $n + 1$, the *successor* of n, is also a positive integer.
- **Axiom 3** Every positive integer other than 1 is the successor of a positive integer.
- **Axiom 4** **The Well-Ordering Property** Every nonempty subset of the set of positive integers has a least element.

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