DISCRETE MATHEMATICS AND APPLICATIONS

Proof Techniques 1

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Chapter Description

Chapter Outline

4.1 Direct Proof

4.2 Indirect Proof

4.3 Contradiction Method

Aims

– Apply direct method to prove a theorem

– Apply indirect method to prove a theorem

– Apply contradiction method to prove a theorem

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References

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Basic Terms

TERM	DESCRIPTION
Theorem	A theorem is a statement that can be shown to be true. Theorems can also be referred as facts or results.
Proof	A proof is a valid argument that established the truth of a theorem. A proof can include axioms (or postulate), which are statements we assume to be true.
Proposition	Less important theorems sometimes are called propositions.
Lemma	A lemma is a less important theorem that is helpful in the proof of other result.
Corollary	A corollary is a theorem that can be established directly from a theorem that has been proved.
Conjecture	A conjecture is a statement that is being proposed to be a true statement.

ODD and EVEN

7 is odd since there exist 3 such that 7=2(3)+1

n is ODD if:

$$\exists k \in \mathbb{Z} \ni n = 2k + 1$$

n is EVEN if:

$$\exists k \in \mathbb{Z} \ni n = 2k$$

100 is even since there exist 50 such that 100=2(50)

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Methods of Proof

To prove the theorem given in implication (if p then q, $p \rightarrow q$), we have three methods of proof:

1
- Direct Method

2
- Indirect Method
- Contrapositive Method

3
- Contradiction Method

Method 1: Direct Method

- Assume the results (theorem etc.) are given in implication proposition logic (if p then q, $p \to q$)

- Direct method needs us to assume p is true and shows that q is true.

Example:

Prove that if x is even, then x^2 is even.

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Direct Method: Example 1

Prove that if x is even, then x^2 is even.

Proof:

- Let

$$p : x \text{ even} \qquad\qquad q : x^2 \text{ even}.$$

- Assume that x is even is true. Need to prove that x^2 is even.
- Since x is even, $x = 2k, k \in \mathbb{Z}$.

  Then

$$x^2 = (2k)^2 = 4k^2 = 2(2k^2) = 2m \quad (m = 2k \in \mathbb{Z})$$

- Therefore x^2 is even. ■

Direct Method: Example 2

Prove that if x and y are odd, then $x+y$ is even.

Proof:

- Let $p: x, y$ odd and $q: x+y$ even.
- Assume that x and y are odd is true. Need to prove that $x+y$ is even.
- Since x and y are odd, then

  $x = 2k+1, y = 2l+1$ where $k, l \in \mathbb{Z}$.
- Thus, $x+y = (2k+1)+(2l+1) = 2(k+l+1) = 2m$

  where $m = k+l+1 \in \mathbb{Z}$.
- Therefore $x+y$ is even. ■

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Direct Method: Example 3 (i)

Prove that if a and b are both perfect squares integer, then ab is also a perfect square integer.

Proof:

- Let p: a and b are both perfect squares integer
  q: ab is also a perfect square integer.
- Assume that a and b are both perfect squares integer is true.
- Need to prove that ab is also a perfect square integer.
- Since a is perfect squares integer, then $\exists x \ni a = x^2$ where $x \in \mathbb{Z}$.
- Since b is perfect squares integer, then $\exists y \ni b = y^2$ where $x \in \mathbb{Z}$.

Direct Method: Example 3 (ii)

Proof:

- Let p: a and b are both perfect squares integer
  q: ab is also a perfect square integer.
- Assume that a and b are both perfect squares integer is true .
- Need to prove that ab is also a perfect square integer.
- Since <u>*a* is perfect squares </u>integer, then $\exists x \ni a = x^2$ where $x \in \mathbb{Z}$.
- Since <u>*b* is perfect squares</u> integer, then $\exists y \ni b = y^2$ where $x \in \mathbb{Z}$.
- Then $ab = x^2 y^2 = (xy)^2 = z^2$ where $z = xy \in \mathbb{Z}$
- Therefore ab is also a perfect square integer. ■

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Method 2: Indirect Method

- It is also known as contrapositive method.
- Contrapositive of implication $p \to q$ is $\sim q \to \sim p$.
- By true table we can show that $p \to q \equiv \sim q \to \sim p$.
- Thus, if it is difficult to prove $p \to q$ by using direct method, we can rewrite and proof by contrapositive $\sim q \to \sim p$ since both are equivalent proposition.

By using indirect method, we need to

(i) assume $\sim q$ is true,

(ii) shows that $\sim p$ is true.

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Indirect Method: Example 1

Prove that if x^2 is odd, then x is odd

Proof:

- Let $p: x^2$ odd and $q: x$ odd.
- By contrapositive method,

  $\sim q: x$ is even and $\sim p: x^2$ is even.
- Assume that x is even is true. Need to prove that x^2 is even.
- The proof follows Example 3 ■

Indirect Method: Example 2 (i)

Give an indirect proof of the theorem
"If $5n+2$ is odd, then n is odd" where n is an integer.

Proof:

- Let

  $p: 5n+2$ is odd and $q: n$ odd.

- By contrapositive method,

  $\sim q: n$ is even and $\sim p: 5n+2$ is even.

- Assume that n is even is true. Need to prove that $5n+2$ is even.

Indirect Method: Example 2 (ii)

Proof (cont.):

- Let $n = 2k,\ k \in \mathbb{Z}$.
- Then

$$\begin{aligned} 5n + 2 &= 5(2k) + 2 \\ &= 10k + 2 \\ &= 2(5k + 1) \\ &= 2l \quad \text{where} \quad l = 5k + 1 \in \mathbb{Z} \end{aligned}$$

- Therefore, $5n + 2$ is even.
- Equivalently shows that if $5n + 2$ is odd, then n is odd. ■

Method 3: Contradiction Method

- Prove $p \rightarrow q$ is true.
- Assume ***p*** and ***~q*** are **true** **(p→~q is true)** and show that ***q*** must also be **true**.

Example:

If $n^2 \equiv 1 \pmod{2}$, then n is an odd integer

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Contradiction Method: Example 1 (i)

Give a proof by contradiction of the theorem
"If $5n+2$ is odd, then n is odd".

Proof:

- Let p: $5n+2$ is odd and q: n is odd
- Assume $p \to \sim q$ is true, which mean if $5n+2$ is odd then n is even.
- Let $n = 2k,\ k \in \mathbb{Z}$. Then

$$\begin{aligned} 5n+2 &= 5(2k)+2 \\ &= 10k+2 \\ &= 2(5k+1) \\ &= 2l \quad \text{where } l = 5k+1 \in \mathbb{Z} \end{aligned}$$

even $(\to\leftarrow)$-contradiction

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Contradiction Method: Example 1 (ii)

Proof:

- Let p: $5n+2$ is odd and q: n is odd
- Assume $p \to \sim q$ is true, which mean if $5n+2$ is odd then n is even.
- Let $n = 2k,\ k \in \mathbb{Z}$. Then

$$\begin{aligned} 5n+2 &= 5(2k)+2 \\ &= 10k+2 \\ &= 2(5k+1) \\ &= 2l \quad \text{where } l = 5k+1 \in \mathbb{Z} \end{aligned}$$

even $(\rightarrow\leftarrow)$-contradiction

- Which is contradiction with our assumption that $5n+2$ is odd.
- Thus, if $5n+2$ is odd, then n is odd is true. ■

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Contradiction Method: Example 2 (i)

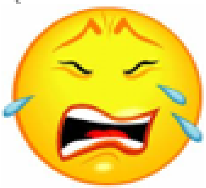
Give a proof by contradiction of the theorem
"If $n^2 \equiv 1 \pmod{2}$, then n is an odd integer".

Proof:

- Let p: $n^2 \equiv 1 \pmod{2}$ and q: n is odd integer
- Assume $p \rightarrow \sim q$ is true, which mean if $\boxed{n^2 \equiv 1 \pmod{2} \text{ is odd}}$ then n is even integer.
- Let $n = 2k,\ k \in \mathbb{Z}$. Then

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 2\left(2k^2\right) \\ &\boxed{\equiv 0 \pmod{2}} \end{aligned}$$

$(\rightarrow\leftarrow)$-contradiction



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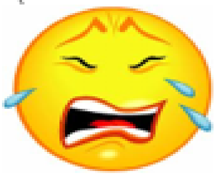
Contradiction Method: Example 2 (ii)

Proof:

- Let p: $n^2 \equiv 1 \pmod 2$ and q: n is odd integer

- Assume $p \rightarrow \sim q$ is true, which mean if $\boxed{n^2 \equiv 1 \pmod 2 \text{ is odd}}$ then n is even integer.



- Let $n = 2k,\ k \in \mathbb{Z}$. Then
$$\begin{aligned} n^2 &= (2k)^2 \\ &= 2\left(2k^2\right) \\ &\boxed{\equiv 0 \pmod 2} \end{aligned}$$
$(\rightarrow\leftarrow)$-contradiction

- Which is contradiction with our assumption that $n^2 \equiv 1 \pmod 2$.

- Thus, if $n^2 \equiv 1 \pmod 2$, then n is an odd integer is true. ■

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